The Power Spectral Density for real-valued functions having a mean-square value

We start from a theorem: under very general conditions, if $V(t)$ is any complex-valued function which is square-integrable, then a Fourier transform $\tilde{V}(f)$ exists, and is given by

$$\tilde{V}(f) = \int_{-\infty}^{\infty} V(t)\, e^{+2\pi i f t}\, dt \quad .$$

The companion inverse transformation is given by

$$V(t) = \int_{-\infty}^{\infty} \tilde{V}(f)\, e^{-2\pi i f t}\, df \quad .$$

Notice that we are using ordinary frequency f (not theorists' angular frequency $\omega = 2\pi f$) in these expressions, and so the 2π factors are needed inside the exponents. But we do *not* need $(2\pi)^{-1}$ or $1/\sqrt{(2\pi)}$ in front of these expressions to normalize them.

With this Fourier-transform pair, we can prove Parseval's theorem, which says

$$\int_{-\infty}^{\infty} \left| V(t) \right|^2 dt = \int_{-\infty}^{\infty} \left| \tilde{V}(f) \right|^2 df \quad .$$

Thus for the absolute-squared functions, the integral over all time of $| V(t) |^2$, or the integral over all frequencies of $| \tilde{V}(f) |^2$, give a single common value. These are the sort of 'normalizable functions' we encounter in 1-d Schrödinger wave mechanics.

But here we turn to a real-valued function $V(t)$, such as a voltage; and now we need to deal with a ever-ongoing function $V(t)$, as in a periodic waveform or stationary noise, which does *not* have a finite value for

$$\int_{-\infty}^{\infty} \left| V(t) \right|^2 dt \quad ,$$

so it is *not* square-integrable. But physically we deal with functions that *do* have a finite mean-square value, such that an average taken over a long time interval, T,

$$\frac{1}{T} \int_{-T/2}^{T/2} \left| V(t) \right|^2 dt$$

does exist and has a limit, denoted $\langle V^2(t) \rangle$, as $T \to \infty$. We call this the 'mean square' of the waveform $V(t)$. So we can write an integral that teaches us we really want the Fourier transform of $V(t)/\sqrt{T}$,

$$\left\langle \left| V(t) \right|^2 \right\rangle = \lim \frac{1}{T} \int_{-T/2}^{T/2} \left| V(t) \right|^2 dt = \lim \int_{-T/2}^{T/2} \left| \frac{V(t)}{\sqrt{T}} \right|^2 dt \quad ,$$

and we write the desired transform as

$$\tilde{W}_T(f) = \int_{-\infty}^{\infty} \frac{V(t)}{\sqrt{T}}\, e^{+2\pi i f t}\, dt \quad .$$

Because of the way $V(t)/\sqrt{\mathrm{T}}$ and $\tilde{W}_T(f)$ are a Fourier-transform pair, it is true from Parseval's theorem that

$$\left\langle V^2(t) \right\rangle = \lim \int_{-T/2}^{T/2} \left| \frac{V(t)}{\sqrt{T}} \right|^2 dt \quad \text{and} \quad \int_{-T/2}^{T/2} \left| \frac{V(t)}{\sqrt{T}} \right|^2 dt = \int_{-\infty}^{\infty} \left| \tilde{W}_T(f) \right|^2 df \quad .$$

On the left, we recognize a quantity which goes to $\langle V^2(t) \rangle$ in the $T \to \infty$ limit. On the right, we first show that a real-valued $V(t)$ implies that $\tilde{W}_T(f)^* = \tilde{W}_T(-f)$, so that $| \tilde{W}_T(f) |$

is an *even* function; and then we can change the integral over positive and negative *f*-values to double the integral over positive frequencies only. That leaves us with

$$\left\langle V^2(t)\right\rangle = \int_{-\infty}^{\infty} \left|\tilde{W}_T(f)\right|^2 df = 2\int_{o}^{\infty} \left|\tilde{W}_T(f)\right|^2 df \quad .$$

And this shows that the mean-square value of $V(t)$, computed in the time domain, can also be fully accounted for in the frequency domain, if we define a 'power spectral density' function

$$S(f) = 2\left|\tilde{W}_T(f)\right|^2 \quad .$$

Here we still need the large-T limit, and there are further details about the sense in which the limit exists. But as defined, our spectral-density function will preserve our accounting-of-energy principle, now written as

$$\left\langle V^2(t)\right\rangle = 2\int_{0}^{\infty} \left|\tilde{W}_T(f)\right|^2 df = \int_{0}^{\infty} S(f)\, df \quad .$$

So the 'energy' (or more properly the power) present in the waveform, in the mean-square-value sense, can be accounted for either by an integral in time, or instead by an integral (over positive frequencies only) of the power spectral density function $S(f)$. This is accordingly called a 'single-sided power spectral density'. A region displaying $S(f) =$ constant is said to describe 'white noise', since two bins of equal width in frequency convey equal powers.

There should be no mystery about the 'positive and negative frequencies' issue, as these arise only due to the use of complex exponentials as a basis. To get a real-valued cosine (or sine) function requires the addition of two complex exponentials, one of positive and the other of negative exponent. So what we mean by 'integrating over positive and negative frequencies' in the complex language is really just 'adding up the effects of both sines and cosine' in ordinary language – and clearly, both sines and cosines are needed to get a full account of all the power.

Notice also the units of his equation. On the left, we have units of Volts-squared. On the right, we have units of *S*, which are V^2/Hz, integrated over frequency, giving Volts-squared again. And those V^2/Hz units for *S* really do follow from the units for the function $W_T^{\sim}(f)$, as can be confirmed from its definition.

Finally, if the function *S* is to be called power spectral density, with units V^2/Hz, then the function $\sqrt{S}$ could also be defined, with units V/√Hz. This is the language in which the size of *S* is usually communicated, because most readers respond better to

$\sqrt{S} = 8$ nV/√Hz

than they do to what defines it, namely

$S = 64 \times 10^{-18}$ V^2/Hz..

Note, however, that it is the function *S* itself, and not $\sqrt{S}$, which obeys the integral condition arising from the Parseval relation. It is also *S* itself, and not $\sqrt{S}$, which has the approximate behavior of $\propto f^{-1}$ in the case of 'excess low-frequency noise' or so-called 1/*f* noise.