Exercise in spectra, and spectral density

In this exercise, we combine the Filtered Noise source you've been using with Amplification and further Filtering, and then we sum it with an externally-generated sinusoid, to produce a waveform of the sort often seen in the lab: a signal polluted by, or even buried under, broadband noise. And then we see how to quantify that signal, and quantify the noise, and how to use the 770 to see that signal amid the noise, in a picture vastly clearer than anything visible in the time domain.

Start with your Buried-Treasure module's Filtered-Noise output, and amplify it by x10 and x2.5 using the Wide-Band Amplifier module. Now further filter that output, by sending it to the input of your Filter module. Set that filter to frequency 20 kHz and Q = 0.71, and look at the Low-Pass output of the filter module. The original noise spectrum has been initially filtered free of very-high-frequency components, then amplified, and now further filtered, basically to have

very nearly the amplified content, for frequencies $0 < f < 20$ kHz,
but rapidly dropping content for frequencies $f$ above 20 kHz.

In fact, the behavior of this filter for white noise is akin to that of a sharp-edged or 'brick-wall' filter, which would multiply the frequency content of the input

by ·1 for $0 < f < 22.2$ kHz,
and by ·0 for $f > 22.2$ kHz.

That is to say, the 'equivalent noise bandwidth' of this particular filter is 22.2 kHz.

Now send the filter's output to the Summer unit; to its other input, send a sinusoid from an external generator. Set that generator to a frequency near 10 kHz, and an amplitude of 0.3 Volts. The output of the summer will resemble this 'scope view:

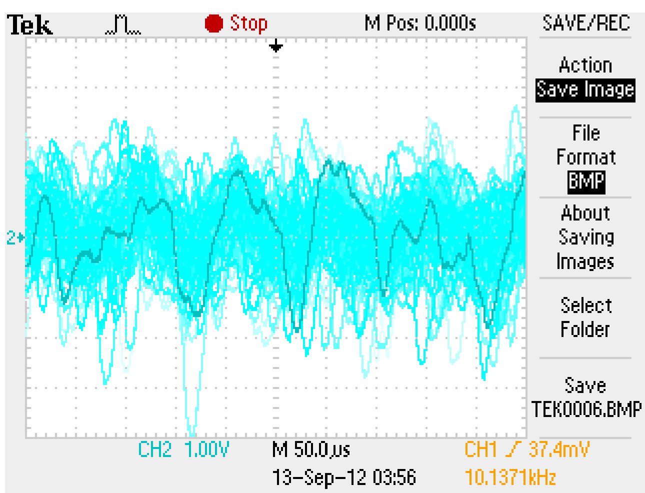


Fig. 6.3: A sinusoidal signal, of period 100 µs and amplitude 0.3 V, well-buried under noise.

This noisy waveform hides a sinusoid of amplitude 0.3 V behind noise that splatters all over the ±2-V range, and beyond. Apply that signal-plus-noise waveform to your 770, and look as follows:

**1:** In full frequency Span, notice the 'noise floor' across the display; because of the filters' actions, it is spectrally flat from 0 to 20 kHz, and then drops monotonically in the 20-100 kHz range. (Try temporarily using a logarithmic scale for the frequency axis to help see this better.) Confirm, by temporarily lowering the Amplifier gain 10-fold, that your noise module really is the source of this noise background. (The noise floor should drop by 20 dB, ie. 100-fold in power, due to 10-fold smaller voltage gain at the amplifier.)

**2:** Be sure you've set the Measure softkey to PSD, and units to Volts rms, and use Averaging to estimate the voltage spectral density of the nose floor. You might find a value near 4 mVrms/√Hz.

**3:** Now look at the spectrum in the vicinity of 10 kHz and spot the 'spectral line' due to your signal generator, lying above the 'floor' of continuous spectrum from the noise source. Temporarily change the frequency, or kill the amplitude, of your generator, to confirm the origin of this feature. Note how prominent the spectral line is, in this frequency-domain view, compared to the signal's total *submergence* under noise in the time-domain view of Fig. 6.3.

**4:** Switch to the appropriate function to quantify this 'signal peak': use the Measure softkey to change from PSD to Spectrum, and (since you're trying to measure the height of a peak) use the Window softkey to change to a Flattop window. To see that you really have a stable rms measure for this signal, now zoom in on the frequency axis, reducing the Span in steps of 2, eventually using a span of 390 Hz or less, centering your signal peak in the display. The rms measure of this signal will scarcely change in the process – particularly as the frequency bins get narrow, and your peak height is minimally contaminated by noise. Even the power spectral density of the noise floor won't change in its Vrms/√Hz units as you do this zooming process – check out that claim also. But Vrms and Vrms/√Hz are literally incommensurate units, and the *prominence* of a signal peak against a noise floor will rise, dramatically, when you zoom in by this method.

[Why? Because all the spectral power in the signal ends up in a single spectral bin, whether that bin is 250 Hz or <1 Hz wide. But the amount of spectral power of the noise part of the waveform which ends up in any particular bin drops *linearly* with the frequency-width of the bin, and you've lowered that bin-width by 256-fold during this exercise.]

[This lunch is not free! Notice that he narrower you make your frequency span, and the more the signal peak stands up above the noise floor, the *longer* the acquisition time which is required.]

**5:** Now learn to predict the rms measure of the signal-plus-noise waveform, using

$$\left\langle [V_{sig}(t)+V_{noise}(t)]^2 \right\rangle = \left\langle V_{sig}{}^2(t)\right\rangle + \left\langle V_{noise}{}^2(t)\right\rangle \quad .$$

(and why is there no cross term in this formula?)

- here $\langle V_{sig}^2(t)\rangle$ is the mean-square measure of the signal, given by the square of its rms measure. (Squaring undoes the 'rooting', leaving just the mean square.)
- here $\langle V_{noise}^2(t)\rangle$ is given instead by an integral over a density, via

$$\int_0^\infty S(f)\, df \approx (S_{noise})\cdot\Delta f = (S_{noise})\cdot(22.2\ kHz)$$

where $S_{noise}$ is the approximately-constant power spectral density you've measured on the flat part of the noise floor, and $\Delta f$ is the equivalent noise bandwidth (22.2 kHz, for this particular filter) over which that $S_{noise}$ value is applicable.

When you've predicted $\langle V^2(t)\rangle$ by combining these results from the 770, ie. when you've done an estimate wholly from observations conducted in the *frequency* domain, find a way to measure the mean-square value of $V(t)$ in the *time* domain. You might find a 'true rms' voltmeter which does the rms measurement correctly, even for signals with frequencies up to >20 kHz; or you might use the 'measure' capability of a 'scope. A voltmeter will effectively average over its update time of half-a-second or so; a 'scope will average over its full-screen acquisition time – you might use 50 μs/div or more on the sweep speed, to give >500 μs of voltage-logging time.

Once either of these tools has given you an rms value, square that number to get a mean-square value $\langle V^2(t)\rangle$ in Volts-squared, derived from time-domain measurements. Now for two questions:

a) Does you time-domain measure *agree* with your frequency-domain estimate?
b) In the sum

$$\left\langle V_{sig}^{\ 2}(t)\right\rangle + \left\langle V_{noise}^{\ 2}(t)\right\rangle \quad ,$$

what fraction of the sum is due to the first, the signal, term?

For the data of Fig. 6.3, that fraction is under 11%, so the waveform is *more than 89% noise* on a power basis. Yet a frequency-domain view of that signal-under-noise, viewed at sufficient spectral resolution, can show a signal peak standing >30 dB, ie. *a thousand-fold in power*, above the local noise floor.

This exercise should give you a vivid picture of why frequency-domain methods are used very broadly for 'signal recovery' in the presence of noise. For more fun with such exercises, see the projects in Ch. 15 of this manual.

**6:** Now you can vary the sizes of signal and noise independently – vary the signal strength with your generator's amplitude adjustment, and the noise level by changing the gain in the broadband-amplifier module. You can independently remove either contribution from the signal-plus-noise superposition by removing one or the other input cable to the summer unit. You can use a true-rms digital voltmeter at the summer's output to quantify the output in rms measure, and you can square that number to give a mean-square value.

For fun, prepare a signal-plus-noise combination with equal (rms) contributions of signal and noise. Perhaps you can adjust each to give 0.7 V (rms). What happens when both waveforms are going into the summer? A look with a ‘scope will show you a waveform with ‘signal-to-noise’ ratio of 1:1, in which you can see the signal by eyeball, despite the noise. But what will the true-rms meter read?

Answer: *not* 0.7 + 0.7 = 1.4 Volts (rms). Instead, do the computation this way: if the square root of the mean square, ie. the rms value, is 0.7 Volts, then upon squaring, you get the plain mean-square, with value $(0.7\ \text{V})^2 = 0.49\ \text{V}^2$. So you have two waveforms, each of which has been set to give a mean-square of $0.49\ \text{V}^2$. And since mean-square values are additive (for these uncorrelated waveforms), you expect $0.49 + 0.49 = 0.98\ \text{V}^2$ as the mean-square measure of the output. The square root of this mean-square is a root-mean-square, rms, value, of 0.99 V. And *that* is what your true-rms meter should read.

Final question: if you’ve prepared a noise signal with dc-to-20 kHz coverage, and if its rms value is 0.7 V, what will its power spectral density be, upon measurement with the 770? Expected answer: the spectrum is flat, ie. ‘white’, from dc upwards, and starts dropping at a ‘corner’ at 20 kHz, decreasing rapidly (though not abruptly) to zero at higher frequencies. We claim that the filtered spectrum acts as if it were flat to 22.2 kHz, and zero thereafter. Now we have a measure of 0.7 V (rms), so squaring, we conclude the waveform has a mean-square value of $0.49\ \text{V}^2$. So its spectral density is

$$S = (0.49\ \text{V}^2) / (22.2 \times 10^3\ \text{Hz}) = 22.1 \times 10^{-6}\ \text{V}^2/\text{Hz} \quad .$$

The square root of this is the voltage spectral density, $4.7 \times 10^{-3}\ \text{V}/\sqrt{\text{Hz}}$ or $4.7\ \text{mV}/\sqrt{\text{Hz}}$. The 770 should show this result when set to units of Volt-rms, indicating 4.7 $\text{mVrms}/\sqrt{\text{Hz}}$. Try it out.