Contrasting multi-sine and random-pulse models of noise

This development addresses two methods for modeling noise waveforms analytically. One is called a 'multisine' method, and the other describes a train of pulses occurring at random times. Each can be a useful way to model noise, and the effect of noise on physical systems; each can have its time-domain and frequency-domain behavior fully understood. Best of all, each can have its spectral-density function $S(f)$ computed exactly.

a) **A multisine representation of noise**

A multisine representation starts with a fundamental period T, which might be rather short in fact, but conceptually could be hours or days, ie. longer than the duration of any given experiment. It then defines a fundamental frequency $f_1 \equiv 1/T$, and considers sinusoidal functions at frequency f_1 and its harmonics. The frequencies involved are $f_i = i \cdot f_1 = i/T$, extending from integer i = 1 up to some chosen highest frequency, the Mth harmonic of f_1. The simplest multisine representation assigns an *equal* amplitude a, and a *random* phase ϕ_i, to each of M sinusoids, and then writes a voltage function of time according to

$$V(t) = \sum_{i=1}^{M} a \cos(2\pi \cdot i f_1 \cdot t - \phi_i) \quad .$$

So the frequency-domain view of this waveform is a 'frequency comb', as there is exactly one sinusoid in each frequency interval $\delta f = f_{i+1} - f_i = 1/T$.

It is easy to work out the mean-square of this waveform, by squaring $V(t)$ and integrating over the full period T. In the process, all the 'cross terms' vanish, leaving

$$\left\langle V^2(t) \right\rangle = \frac{1}{T}\int_{-T/2}^{T/2} V^2(t)\,dt = M \cdot \frac{a^2}{2} \quad .$$

It is also easy to assign the spectral density of this waveform, by imagining first filtering the waveform to some frequency band of width Δf at frequency f. Such a band-pass filtering will let through not all M terms, but only a smaller number, namely $\Delta f/\delta f$, of terms in the sum above, and the mean-square value of the *filtered* waveform is then

$$\left\langle [V^2(t)]_{\text{filtered}} \right\rangle = \frac{\Delta f}{\delta f} \cdot \frac{a^2}{2} = \frac{a^2}{2\delta f}\Delta f \quad .$$

Now we compute the mean-square per unit bandwidth, giving

$$S(f) = \frac{\left\langle [V^2(t)]_{\text{filtered}} \right\rangle}{\Delta f} = \frac{a^2}{2\delta f} \quad .$$

Thus we see that the power spectral density $S(f)$ is given by

$$S(f) = \frac{a^2}{2\delta f} \quad .$$

at every frequency up to the highest frequency present in the sum (provided that we do not look with sufficient spectral resolution to see the individual teeth in the frequency comb).

By contrast to the simple frequency-comb form that this model displays in the *frequency* domain, its appearance in the *time* domain can be complicated. In fact, unlike the power spectrum, the time-domain waveform depends in detail on the choices made for the phases ϕ_i. If the phases are all chosen to be zero, $V(t)$ adds up to a series of narrow spikes, occurring periodically in time with period T. If the phases are chosen instead according to certain deterministic laws (eg. Schroeder phases), then $V(t)$ can take on the form of a 'chirp' waveform. If the phases are picked to be *random* in the interval $(0, 2\pi)$, then the waveform takes on the familiar look of noise, and in fact displays near-Gaussian statistics in its voltage histogram. But for *any* of these time waveforms, the frequency-domain power spectrum is either

- a frequency comb with tooth spacing δf, when seen with high enough spectral resolution, or
- effectively a white-noise continuum, of spectral density $S = a^2/(2\ \delta f)$, when seen with lower resolution.

b) **A random-pulse model of noise**

We now introduce an alternative representation of noise. This model starts with narrow pulses in time, and it requires that we assume pulses occur at randomly-distributed times. In principle this pulse train goes on eternally, but we imagine instead that the pulses exist only in the time interval $-T/2 < t < T/2$. Or, we could imagine 'windowing', out of a continuous train of pulses, only those pulses which fall in an observation window of duration T. We could imagine that the 'true' waveform is periodic with period T, or that the waveform vanishes outside this window of width T, but in any case we can again choose T to be longer than the duration of any plausible experiment, so in practice it does not matter.

So inside our time window, we imagine that N pulses occur, and the times of occurrence are labeled t_j, where j runs from 1 to N. At each of these times, we imagine a voltage pulse, which we model to be rectangular in shape, of height H in voltage, and duration τ in time. [We postpone the discussion of that limiting case in which the pulses are modeled as delta-functions.]

So our waveform has squared-value either H^2 or 0, and its mean-square value is easily computed (provided we assume the pulses are narrow enough, or infrequent enough, that pulse overlap can be neglected):

$$\left\langle V^2(t) \right\rangle = \frac{1}{T}\int_{-T/2}^{T/2} V^2(t)\,dt = \frac{1}{T}\cdot N \cdot H^2\,\tau \quad .$$

With a view to a future random pulse train, we let the quotient $N/T \equiv r$, the rate of arrival of pulses.

This calculation shows that our waveform is square-integrable, so it has a Fourier transform. In fact, according to the treatment of Appendix A.10 of 'Noise Fundamentals', we can even write a candidate for its power spectral density, in the form

$$S(f) = 2\left|\tilde{W}_T(f)\right|^2 \quad ,$$

where the scaled Fourier transform is given by

$$\tilde{W}_T(f) = \frac{1}{\sqrt{T}}\int_{-T/2}^{T/2} V(t)\, e^{2\pi i f t}\, dt \quad .$$

From this definition we get a power spectral density that obeys a form of the Parseval relation

$$\int_0^\infty S(f)\, df = \left\langle V^2(t)\right\rangle \quad .$$

Now using the function $R(t)$ to represent a rectangular pulse of unit height, and fixed width τ, centered on the origin, our waveform is given by

$$V(t) = \sum_{j=1}^{N} H\ R(t - t_j) \quad .$$

With this form of $V(t)$, the transform W is also easily computed, and gives

$$\tilde{W}_T(f) = \frac{1}{\sqrt{T}}\int_{-T/2}^{T/2} e^{2\pi i f t}\sum_{j=1}^{N} H\ R(t - t_j)\, dt \quad .$$

Clearly we can interchange the order of integration and summation. Then in each of N integrals that appear, we can let $t' = t - t_j$, and each term can be manipulated to give

$$\int_{-T/2}^{T/2} e^{2\pi i f t}\, R(t - t_j)\, dt = \int_{-\tau/2}^{\tau/2} e^{2\pi i f (t' + t_j)}\, R(t')\, dt' \quad .$$

The integral that's left is easily done, and it gives

$$\int_{-\tau/2}^{\tau/2} e^{2\pi i f (t' + t_j)}\, R(t')\, dt' = e^{2\pi i f t_j}\int_{-\tau/2}^{\tau/2} e^{2\pi i f t'}\cdot 1\cdot dt' = e^{2\pi i f t_j}\,\frac{1}{2\pi i\, f}\cdot 2 i \sin(\pi f \tau)$$

$$= e^{2\pi i f t_j}\cdot \tau \cdot \mathrm{sinc}(\pi f \tau) \quad .$$

So our W-function becomes

$$\tilde{W}_T(f) = \frac{H}{\sqrt{T}}\sum_{j=1}^{N} e^{2\pi i f t_j}\cdot \tau\, \mathrm{sinc}(\pi\ f\ \tau) \quad ,$$

where we note that the $\mathrm{sinc}(x) \equiv (\sin x)/x$ factor is a result of our assumption of rectangular pulses. Since this factor occurs in every term of the sum, we can write

$$\tilde{W}_T(f) = \frac{H\tau}{\sqrt{T}}\,\mathrm{sinc}(\pi\ f\ \tau)\sum_{j=1}^{N} e^{2\pi i f t_j} \quad .$$

The sinc-function has its first zero at $(\pi f \tau) = \pi$, ie. at frequency $f = 1/\tau$, so the computed noise spectrum will be flat near $f = 0$, and will decrease when approaching a frequency of order $1/\tau$.

The power spectral density is now given by

$$S(f) = 2\left|\tilde{W}_T(f)\right|^2 = 2\frac{H^2\tau^2}{T}\mathrm{sinc}^2(\pi f \tau)\left|\sum_{j=1}^{N} e^{2\pi i f t_j}\right|^2 \quad .$$

The squared-sum can be written as

$$\sum_{j,k=1}^{N} (e^{2\pi i f t_j})(e^{2\pi i f t_k})^* \quad ,$$

and in the sum we can separate the ‘diagonal terms’ with $j = k$ from the ‘off-diagonal’ terms with $j \neq k$. That gives for the sum the result

$$\sum_{j=1}^{N} e^{2\pi i f \cdot 0} + \sum_{j \neq k} e^{2\pi i f (t_j - t_k)} \quad .$$

The first part has N occurrences of 1, so it sums up to N. The second part has $N^2 - N$ terms, but these occur in pairs, and the result can be written in manifestly real form as

$$N \cdot 1 + \sum_{j>k} 2 \cdot \cos[2\pi\, f\, (t_j - t_k)] \quad .$$

Hence the spectral density is

$$S(f) = 2\frac{H^2\,\tau^2}{T}\operatorname{sinc}^2(\pi f \tau)\left\{N + \sum_{j>k} 2\cos[2\pi (t_j - t_k)\, f\,]\right\} \quad .$$

Apart from an overall multiplier and a sinc^2-factor, the spectral density includes within brackets a constant, plus the sum of $N(N\text{-}1)/2$ cosines-in-frequency. The periodicity-in-frequency of a generic term is $1/(t_j - t_k)$, which is typically of order $2/T$. Since as T is imagined to be long, these terms vary rapidly in frequency, and give the computed $S(f)$-function its rapid fluctuations with respect to frequency. But the mean (with respect to frequency) of each such cosine term is zero, so the local average-in-frequency of spectral density is

$$\bar{S}(f) = 2\frac{H^2\,\tau^2}{T}\operatorname{sinc}^2(\pi f \tau)\left\{N + \frac{N(N-1)}{2}\cdot 2 \cdot 0\right\} \quad .$$

Again, the quotient $N/T \equiv r$ emerges, so we finally have the desired result: for a train of rectangular pulses, each of height H and width τ, arriving randomly in time but at average rate r, the power spectral density’s average value is given by

$$\bar{S}(f) = 2\, r\, (H\,\tau)^2 \operatorname{sinc}^2(\pi f \tau) \quad .$$

c) **The delta-function limit?**

It is sometimes useful to think of our voltage pulses in the limiting case of very high and very narrow: $H \to \infty$, $\tau \to 0$, but with ‘area’ product $H{\cdot}\tau$ held fixed at A. Under this assumption, the waveform-in-time is modeled as a train of delta-functions, given by

$$V(t) = \sum_{i=1}^{N} A\,\delta(t - t_j) \quad .$$

In this limit, we get an averaged-in-frequency spectral-density function

$$\bar{S}(f) = 2\, r\, (A)^2 \operatorname{sinc}^2(\pi f\, 0) = 2\, r\, A^2 \quad ,$$

and in our $\tau \to 0$ limit, the spectrum is ‘flat’ or white in frequency, out to $f = \infty$.

This delta-function limit is convenient to use, but it does have one pitfall. It gives $S(f)$ = constant, so the integral

$$\int_0^\infty S(f)\, df$$

which ought to give $\langle V^2(t)\rangle$, the mean-square-value of the voltage, now is found to diverge. For that matter, the earlier direct computation of $\langle V^2(t)\rangle = (N/T)\, H^2\, \tau = r\, H^2\, \tau =$

$r\,(H\,\tau\,)^2/\tau\; = r\,A^2/\tau$ is also seen to diverge as $\tau \rightarrow 0$. So we cannot retain a square-integrable function if we 'go all the way to $\tau \rightarrow 0$', though we may be able to model our pulses as delta-functions for some other mathematical purposes.

Short of the $\tau \rightarrow 0$ limit (as genuine physical pulses always are), we can compute

$$\int_0^\infty \overline{S}(f)\,df = \int_0^\infty 2\,r\,(H\,\tau)^2\,\mathrm{sinc}^2\,(\pi\,f\,\tau)\,df = 2\,r\,(H\,\tau)^2\,\frac{1}{\pi\,\tau}\cdot\frac{\pi}{2} = r\,H^2\,\tau \quad ,$$

and we get a result in exact agreement with the time-domain computation of $\langle V^2(t)\rangle$. This is a welcome check, but not a surprise, as this result of the Parseval relation was built into our definition for $S(f)$.

d) The fluctuations in the spectral density *S*(*f*)

Now that we have a computation of the spectral density, we can even discuss the fluctuations, with respect to frequency f, of $S(f)$'s value, which always show up in a plot of $S(f)$ vs. f for a noise waveform. Those fluctuations arise from the behavior of the bracketed quantity

$$\left\{ N + \sum\nolimits_{j>k} 2\cos[2\pi\;(t_j - t_k)\,f\,] \right\} \quad ,$$

and we've previously noted the first term has a mean-in-frequency of N, while the second term has a mean-in-frequency of 0. Be we can also discuss the mean-square-*fluctuations* in frequency of these two pieces. The first term has no fluctuations; in the second, we assume we are not looking right at $f = 0$, and we assume that the time intervals $(t_j - t_k)$ that appear are 'random enough' that the cosine terms are all of uncorrelated periodicities-in-frequency. Then the mean square of this zero-on-average sum is easily computed to be

$$\frac{N(N-1)}{2}\cdot 2^2\cdot\frac{1}{2} \quad ,$$

since there are $N(N\text{-}1)/2$ terms, each involving a 2^2 and a cosine-squared whose mean-square value is 1/2.

So in the bracket we have two terms, which we can write as

$$\{\quad\} = \{\,[\text{a term of mean of } N, \text{ with mean-square fluctuation } 0] +$$
$$[\text{a term of mean of } 0, \text{ with mean-square fluctuation } N(N\text{-}1)]\;\} \quad ,$$

That is to say, as a function of frequency f, the spectral density $S(f)$ is a function whose mean-in-frequency is $\propto N$, and whose *root*-mean-square measure of fluctuations-in-frequency are $\propto \sqrt{N(N\text{-}1)}$.

This substantiates the claim that the computed version of $S(f)$ from any single capture of a noise waveform will yield a result with large fluctuations in frequency. In fact, we now see that the fluctuations, measured in the rms sense, are just as large as the mean value. These fluctuations do *not* get any smaller, relative to the non-zero mean value of $S(f)$, either as $T \rightarrow \infty$ or as $N \rightarrow \infty$. But it should be clear that local averaging with respect to frequency, or the use of repeated acquisitions to form an ensemble average, will suppress these fluctuations relative to the mean.

e) Shot noise

In any case, we finally have a model for noise which is exactly applicable to a physically-relevant situation. We can write a current composed of randomly-arriving electrons by

$$i(t) = \sum_{i=1}^{N} (-\mathrm{e})\, \delta(t - t_j) \quad .$$

where the times-of-arrival t_j are randomly distributed, but occur at rate r. Then such a current displays 'shot noise', in that its power spectral density is

$$\bar{S}_i(f) = 2\, r\, (-\mathrm{e})^2 = 2\, r\, \mathrm{e}^2 \quad ,$$

Since the magnitude of the d.c.-averaged current i_{dc} is given by $r \cdot \mathrm{e}$, this can also be written as

$$\bar{S}_i(f) = 2\,(r\,\mathrm{e})\,\mathrm{e} = 2\, i_{dc}\, \mathrm{e} \quad ,$$

or in the form $S(f)\, \Delta f = 2\ \mathrm{e}\ i_{dc}\ \Delta f$, the usual expression for the spectral density of shot noise. We can now even see that a real current, composed of pulses of width $\tau \approx 1$ ns, will in practice exhibit a shot-noise spectrum which is 'white in frequency', at least for frequencies well below $1/\tau \approx 1$ GHz.

e) The 'true' picture of noise?

We can now see that a noise waveform with an adequately flat-in-frequency spectral density of noise $S(f)$ can be written in two independent ways. Both methods allow us to write a function-in-time whose spectral density is analytically computable in terms of the parameters of the waveform. In each case, it is clear what we need to do to get a noise spectrum that is flat out to some target frequency. The more interesting question is, what is the actual waveform of some voltage whose noise spectrum we quantify, and verify as white? The answer is that we cannot know, at least from the mere measurement of the spectral density.

This should be clear from the fact that we can construct *two* completely differentnoise waveforms, each adequately flat from $f = 0$ to $f = 1$ MHz, and each delivering spectral density $S(f) = 10^{-12}\ \mathrm{V}^2/\mathrm{Hz}$ in that range (so that the voltage spectral density $\sqrt{S}$ is given by $1\ \mu\mathrm{V}/\sqrt{\mathrm{Hz}}$ in each case). These two waveforms are very different in character. The multisine model would (under sufficient resolution in frequency) reveal itself to be a frequency comb, and would (examined for a long enough duration) reveal itself to be periodic in time. The random-pulse waveform would (under sufficient time resolution) reveal the actual and individual pulses that make it up.

But both give the same spectral density of noise, and both are equally useful representations of noise. For example, either one could be imagined as the excitation applied to a simple harmonic oscillator. The response of the oscillator would be computed differently for the cases of the two waveforms. In the multisine model, we would use the principle of linearity and the transfer function for each individual frequency of sinusoidal excitation. In the random-pulse model, we would use the impulse-response function of the oscillator, and also use linearity. In either case, we would extract an oscillator response, and we would find (for example) the same result from the mean-square value of the oscillator response.

This should illustrate that there are many models for noise, and that the two we have picked are only examples chosen for their simple analytic properties.