Students typically encounter the Fourier series for a periodic function in a context which does not mention sampling, and which gives results looking superficially distinct from the Discrete Fourier Transform we have discussed above. This section is intended to show the very close connection between these two kinds of results.

Suppose, as usual, that we start with a function $V(t)$ which is periodic, with period T. Then by Fourier's Theorem, it has an expansion in the sine and cosine functions which share this periodicity. Alternatively, it has an expansion in the complex exponentials of period T, which can be written as

$$\exp(-2\pi i \frac{k}{T} t) \quad , \quad \text{where } k \text{ is an integer} \quad .$$

Here the term indexed by k has a frequency $k/T = k\cdot(1/T) = k f_1$, where $f_1 = 1/T$ is the fundamental frequency corresponding to a period T. Note that $k = 0$ gives a constant, or 'dc' term; and note that sums over positive and negative k-values will be necessary to make a real-valued expansion out of these complex-exponential 'basis functions'.

Now using this basis, the general periodic function has an expansion involving all integers k, $-\infty < k < \infty$, which can be written as

$$V(t) \; = \; \sum_{k=-\infty}^{+\infty} b_k \; \exp(-2\pi i \frac{k}{T} t) \quad .$$

To extract the value of the coefficient we've written as b_k, we use the usual method: we multiply both sides by $\exp(+2\pi\, i\, (m/T)\, t)$, where m is a free integer, and we integrate over one full period. This gives

$$\int_0^T dt \, \exp(+2\pi i \frac{m}{T} t) \, V(t) = \int_0^T dt \, \exp(+2\pi i \frac{m}{T} t) \sum_{k=-\infty}^{+\infty} b_k \, \exp(-2\pi i \frac{k}{T} t) \quad .$$

If we can interchange the order of summation and integration on the right, we get

$$\int_0^T dt \, V(t) \exp(+2\pi i \frac{m}{T} t) = \sum_{k=-\infty}^{+\infty} b_k \int_0^T dt \, \exp(+2\pi i \frac{m-k}{T} t) \quad .$$

The integral on the right is easy: when m and k are distinct integers, the result is zero, but when $m = k$, the result is just T. So the integral can be written as $T\cdot\delta_{mk}$, and this Kronecker-delta then collapses the (generally infinite) sum to just a single term:

$$\int_0^T dt \, V(t) \exp(+2\pi i \frac{m}{T} t) = \sum_{k=-\infty}^{+\infty} b_k \cdot T \; \delta_{mk} \; = T \cdot b_m \quad .$$

This gives an explicit result for coefficient b_k as

$$b_k = \frac{1}{T} \int_0^T dt \, V(t) \exp(+2\pi i \frac{k}{T} t) \quad .$$