Chapter 0: Fourier Methods – An Overview

What is, and what are, 'Fourier Methods'?

'Fourier Methods' is the name TeachSpin has chosen for a new experimental package, offering special capabilities for understanding waveforms of electronic signals. This package is a combination of the SR770 Fourier Analyzer instrument from Stanford Research Systems, plus the 'Electronic Modules' instrument case, and three external-project experiments from TeachSpin, together with the TeachSpin curriculum you're reading.

> In addition to this Manual from TeachSpin, SRS provides an Operating Manual that came with your SR770 instrument. Boxes like this one in the TeachSpin manual will refer you to resources in the SR770's operating manual. In that grey binder, you may find the section 'Getting Started' most useful; see also the 'troubleshooting' ideas at page 1-47.

More broadly, Fourier methods are a way of thinking differently about signals which vary in time, analyzing them *not* by time-of-occurrence of features, but instead by their *frequency content*. To a 'time-domain view' suggested by a graph of the function $V(t)$, Fourier methods adds the mental, and also the practical, capability of seeing a new graph, a plot which shows the composition of the same signal as a function of frequency. This new graph is another view, the *spectrum*, or the 'frequency-domain view', of the signal.

To be perfectly concrete about this, have a look at an actual $V(t)$ plot in Fig. 0.1 – it's the sort of signal you'll collect in Chapter 13, from a 'fluxgate magnetometer'. The plot of few milliseconds' worth of electrical signal looks like complete garbage – if you got this signal in the lab, you'd suspect a loose connection or a failed instrument.

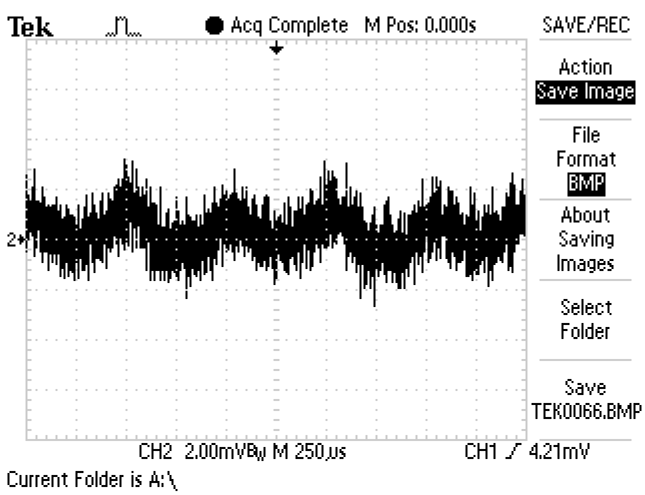

Fig. 0.1: The output signal of a fluxgate magnetometer in normal operation. Horizontal scale is time, 0.25 ms/division; vertical scale is voltage $V(t)$, 2 mV/division.

But now look at the Fourier spectrum of the very same signal, in Fig. 0.2.

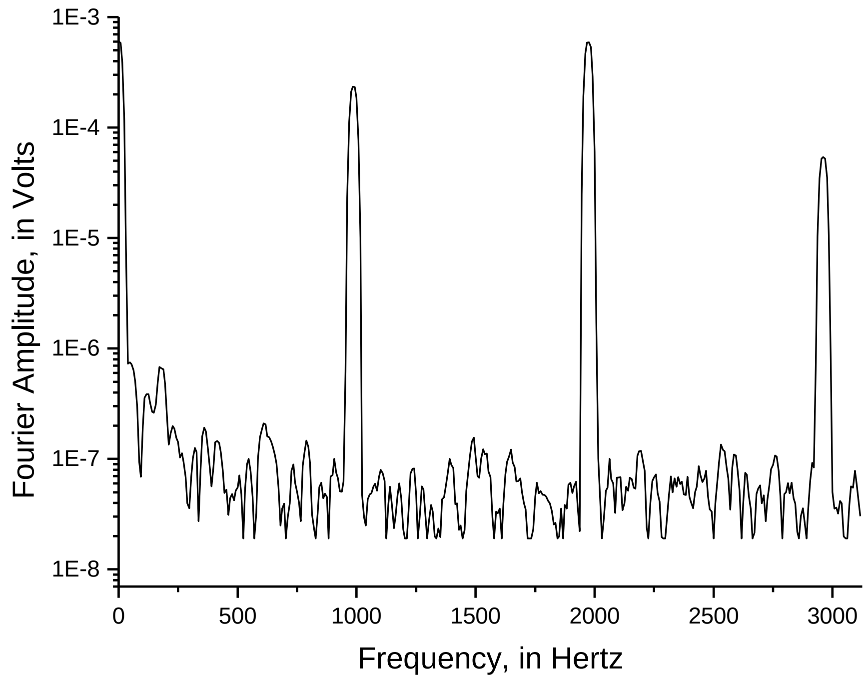

Fig. 0.2: The spectrum of the same signal displayed in the previous figure. Horizontal scale is frequency, 0 to 3.125 kHz; vertical scale is logarithmic, and displays the magnitude of spectral frequency components.

In this display, you can see three frequencies at which there are prominent peaks – much more prominent than you might first think, given that the vertical scale is logarithmic. (There's also something happening down near zero frequency, or 'at d.c.'.) The information content of this signal, wholly obscured in the first figure, is wholly apparent in the second figure. (Better still, you'll learn that the height of the second peak in the spectrum is a direct measure of the static magnetic field which the fluxgate is measuring, and with very attractive sensitivity too.)

There are a host of examples of this character within physics, and many more come from the fields of engineering, communication, and information science. **In so many cases, it is the Fourier spectrum of a signal which makes its information content clear.** A voltage-vs.-time plot of the signal emerging from a radio antenna would look even uglier than Fig. 0.1, in part because the signal is the joint result of radio emissions from a multitude of sources. But if you've 'tuned the dial' of a transistor radio, you have selected *on the basis of frequency* the (unique) component you care to receive from the mishmash of signals arriving at the radio. So it's the Fourier spectrum of the signal – its depiction as a function of frequency, and its de-composition or resolution on the basis of frequency – which reveals the information content, and the usefulness, of the signal.

A first example in Fourier thinking: the Fourier series of a periodic signal

Suppose we have a signal represented as a voltage in time, $V(t)$, and one that it is *periodic*, with some period T. Then you may know that there is a Fourier-series representation of $V(t)$, in the form of a finite or infinite series of sinusoids, which we'll write as

$$V(t) = V_{dc} + \sum_{n=1}^{\infty} [C_n \cos(2\pi \cdot n \cdot \frac{t}{T}) + S_n \sin(2\pi \cdot n \cdot \frac{t}{T})] \quad .$$

This shows that the information content in $V(t)$ can alternatively be represented by a list of constants $\{V_{\mathrm{dc}}, (C_1, S_1), (C_2, S_2), \ldots\}$. That equality sign claims that to know $V(t)$ is to know the list of constants, and that to know the list of constants is to know the signal $V(t)$.

Appendices A1 and A11 show exactly how those *C*- and *S*-constants can actually be computed from the signal $V(t)$, but here's an interpretation of them:

V_{dc} is the time-averaged value of $V(t)$, the 'dc average' of the signal

C_{n}, S_{n} are coefficients of the cosine and sine terms of frequency $f_{\mathrm{n}} = n \cdot f_1 = n\,(1/T)$.

We would say that $V(t)$ consists of a d.c. (or zero-frequency) term, plus 'harmonics' labeled by n:

$n = 1$ labels the fundamental, the cosine and sine terms of frequency $f_1 = 1/T$;

$n = 2$ labels the 'second harmonic', terms of frequency $f_2 = 2 \cdot f_1 = 2\,(1/T)$;

$n = 3$ labels the 'third harmonic', etc.

[Note that in some acoustical and musical contexts, the physicist's 'second harmonic' is called the musician's 'first overtone', ie. the first tone over the fundamental's frequency.]

So if we have a method for determining the coefficients C_{n}, S_{n}, then we have a way to measure the spectrum of the signal. Note too that we often have occasion to combine C_{n} and S_{n} into two new parameters, the magnitude M_{n} and phase φ_{n} of the *n*th Fourier component, according to

$$M_n = \sqrt{C_n^{\,2} + S_n^{\,2}} \quad , \quad \varphi_n = \tan^{-1}(S_n / C_n) \quad \text{or} \quad C_n = M_n \cos\varphi_n \quad , \quad S_n = M_n \sin\varphi_n \quad .$$

Clearly $(C_{\mathrm{n}}, S_{\mathrm{n}})$ constitute an ordered pair in a Cartesian plane, and M_{n} and φ_{n} are a polar representation of the same information.

In what sense is $V(t)$ *equal* to its Fourier representation? There are detailed mathematical proofs laying out the conditions under which the Fourier series converges to the function $V(t)$, and there is the fine point that while the series does converge, it does not converge uniformly. Independent of mathematical niceties, it's certainly feasible to show that, in the limit of an infinite number of terms in the sum, *all* the power in $V(t)$ is fully accounted for in the sum.

Thinking instead more figuratively, you might think of the cosine and sine terms in the series as the 'atoms' out of which the series can be built up, in the sense that the most general periodic signal $V(t)$ can be thought of as a 'molecule' made up of a combination of atoms. Just as 'chemical analysis' is taking a molecule apart into its constituent atoms, so Fourier *analysis* is taking a signal apart into its constituent sinusoidal 'atoms'. And as 'chemical synthesis' is to build a molecule up out of atoms, so Fourier *synthesis* is to build up a waveform out of sinusoids.

There's a very concrete way in which our signal can be viewed from both time-domain and frequency-domain viewpoints, and that is to compute the power in the signal. Applying a voltage $V(t)$ to a resistance R, we get

instantaneous power $P(t) = [V(t)]^2 / R$,

and average power $P_{\text{avg}} = < [V(t)]^2 / R >$,

where the <…> notation stands for taking the time average. But using the Fourier-series representation of $V(t)$ above, it's feasible to show that the average power can also be written as

$$P_{avg} = \frac{V_{dc}^{\ 2}}{R} + \sum_{n=1}^{\infty} \frac{C_n^{\ 2} + S_n^{\ 2}}{2R} \quad .$$

Notice that there is *no* appearance of anything time-dependent in this expression. Here the average rate of energy transport in the signal is attributable to 'power in the dc term', *plus* the power that can be thought of as separately contained in each of the harmonics, with

$$P_n = \text{power in } n\text{th harmonic} = \frac{(C_n^{\ 2} + S_n^{\ 2})/2}{R} = \frac{M_n^{\ 2}}{2R} \quad .$$

Thus the total power in the signal can be dis-aggregated, either by *when* it occurs in time, or *where* it occurs in frequency. These are the time-domain and frequency-domain views of the same power, carried in the same signal. Note too that the power in a signal is fully determined by the magnitudes, and is *in*dependent of the phases, of the Fourier coefficients of its expansion.