Savitsky-Golay Filters

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References

1. Abraham Savitsky and Marcel J. E. Golay, "Smoothing and differentiation of data by simplified least squares procedures," Anal. Chem. **36**, 1627–1639 (1964).

2. Manfred U.A. Bromba and Horst Ziegler, "Application hint for Savitsky-Golay digital smoothing filters," Anal. Chem. **53**, 1583–1586 (1981).

3. Horst Ziegler, "Properties of digital smoothing Polynomial (DISPO) filters," Appl. Spec. **35**, 88–92 (1981).

Introduction

Savitsky-Golay filters permit quick and easy data smoothing and determination of derivatives at each point in a set of data equally spaced in the abscissa. In our introductory physics labs they are used for real time smoothing and evaluation of the velocity and acceleration of objects detected with a motion sensor. They are also used in our advanced physics lab to smooth the angular position and determine the angular velocity and acceleration of a pendulum attached to a rotary encoder.

Here, the input data is considered to be a a set of y-values $y_i, i = 1...N$ equally spaced in time with the time spacing given by τ. The filters are based on a polynomial fitting model to a subset of the data and the calculated derivatives (zeroth is considered smoothing) are determined from the polynomial coefficients for that subset. The order of the polynomial and the number of data points used in the fit affect the determination of the filters. The timing of the data used in the fit relative to the particular point in time at which the smoothed values and derivatives are sought also affects the filters. The timing may be forward-looking (with the particular point at or near the end of the data points), backward-looking (with the particular point at or near the beginning of the data points), or centered (with the particular point at or near the middle of the data points). In fact, any relative timing can be used.

The Savitsky-Golay filter for determining a desired derivative (zeroth is smoothing) is a vector whose size is the same as the number of consecutive y-values to be used in the determination and whose dot product with those y-values provides the desired derivative. The filter values are calculated once and do not change as the particular point is varied. For example, the coefficients for a centered, five-point filter to calculate the second derivative (acceleration a) using a cubic polynomial is given by

$$\{2, -1, -2, -1, 2\}/7\tau^2$$

Multiplying these coefficients by the actual y-values and summing then gives the acceleration at the time of the middle y-value. For example,

$$a(t_6) = (2y_4 - y_5 - 2y_6 - y_7 + 2y_8)/7\tau^2$$

where t_6 is the time at y_6. The set of y-values need only be shifted up one to get the acceleration at the time of y_7.

$$a(t_7) = (2y_5 - y_6 - 2y_7 - y_8 + 2y_9)/7\tau^2$$

Filter construction

The derivation to follow shows how the filters are determined for the case of centered data with one point at the desired time and an equal number of additional data points to each side. The variations needed for other timings should be fairly obvious. For any timing, the origin of time would be adjusted so that the desired time becomes $t = 0$ and its y-value relabeled y_0. This y-value, together with M additional y-values to each side will be used to determine best estimates of y, dy/dt, and d^2y/dt^2 at $t = 0$. The input data set y_m will then be subscripted by the indices $m = -M, -M+1, ..., -1, 0, 1, ..., M-1, M$.

A polynomial fitting model is used: $y(t) = a_0 + a_1 t + a_2 t^2 + ...$ up to order R. That is,

$$y(t) = \sum_{r=0}^{R} a_r t^r \tag{1}$$

The χ^2 is given by

$$\chi^2 = \frac{1}{\sigma_y^2} \sum_{m=-M}^{M} (y(t_m) - y_m)^2 \tag{2}$$

where

$$t_m = -M\tau, ... - 3\tau, -2\tau, -\tau, 0, \tau, 2\tau, 3\tau..., M\tau \tag{3}$$

(i.e., $t_m = m\tau$) and σ_y is the standard deviation of the measured y_m, assumed to be constant. The best estimates of the a_r are then determined by a least squares fit, and the sought-after nth derivatives at $t = 0$ are then given by

$$\frac{d^n y}{dt^n} = n! a_n \tag{4}$$

The set of least squares equations $d\chi^2/da_n = 0$ can be rewritten in the vector-matrix form

$$\mathbf{Y} = [\mathbf{X}]\mathbf{a} \tag{5}$$

where the elements of the column vector $\mathbf{a}$ are the $R+1$ fitting coefficients a_r, $\mathbf{Y}$ is another column vector of $R+1$ elements given by

$$Y_r = \sum_{m=-M}^{M} y_m t_m^r \tag{6}$$

and $[\mathbf{X}]$ is an $R+1$ by $R+1$ square matrix with elements

$$[\mathbf{X}]_{nr} = \sum_{m=-M}^{M} t_m^{n+r} \tag{7}$$

The vector $\mathbf{a}$ is then determined by finding $[\mathbf{X}]^{-1}$, the inverse of the matrix $[\mathbf{X}]$ so that

$$\mathbf{a} = [\mathbf{X}]^{-1}\mathbf{Y} \tag{8}$$

Moreover, the covariance matrix for the parameter estimates, $[\boldsymbol{\sigma}_a^2]$ is given in terms of this inverse matrix

$$[\boldsymbol{\sigma}_a^2] = \sigma_y^2[\mathbf{X}]^{-1} \tag{9}$$

Expressing all elements of Eq. 8 explicitly gives

$$a_r = \sum_{n=0}^{R}[[\mathbf{X}]^{-1}]_{rn} Y_n \tag{10}$$

and substituting Eq. 6 for Y_n

$$a_r = \sum_{n=0}^{R}\sum_{m=-M}^{M}[[\mathbf{X}]^{-1}]_{rn} y_m t_m^n \tag{11}$$

Rearrange to get

$$a_r = \sum_{m=-M}^{M}\left(\sum_{n=0}^{R}[[\mathbf{X}]^{-1}]_{nr} t_m^n\right) y_m \tag{12}$$

Consider the y_m-values as a column-vector $\mathbf{y}$ of $2M+1$ elements. The Savitsky-Golay filters (for the polynomial coefficients) can then be represented as a matrix $[\mathbf{c}]$ having $R+1$ rows and $2M+1$ columns with elements given by the term in enclosed in parentheses above

$$[\mathbf{c}]_{rm} = \sum_{n=0}^{R}[[\mathbf{X}]^{-1}]_{rn} t_m^n \tag{13}$$

so that Eq. 12 for the column vector $\mathbf{a}$ now becomes

$$\mathbf{a} = [\mathbf{c}]\mathbf{y} \tag{14}$$

The t_m are known ahead of time so the matrix $[\mathbf{c}]$ can be predetermined. To do so, first define $[\mathbf{m}]$ as a matrix of $R+1$ rows by $2M+1$ columns having elements

$$[\mathbf{m}]_{rm} = m^r \tag{15}$$

i.e., the explicit form

$$[\mathbf{m}] = \begin{bmatrix} 1 & 1 & ... & 1 & 1 & 1 & ... & 1 & 1 \\ -M & -M+1 & ... & -1 & 0 & 1 & ... & M-1 & M \\ \vdots & \vdots & ... & \vdots & \vdots & \vdots & ... & \vdots & \vdots \\ -M^R & -(M+1)^R & ... & -1 & 0 & 1 & ... & (M-1)^R & M^R \end{bmatrix} \tag{16}$$

With this matrix, it is then easy to show that the term t_m^n can be represented as the following matrix element

$$t_m^r = [[\mathbf{U}][\mathbf{m}]]_{rm} \tag{17}$$

where $[\mathbf{U}]$ is a square diagonal matrix representing the time units, i.e., having only nonzero elements for $[\mathbf{U}]_{nn} = \tau^n$, for $n = 0...R$.

Then Eq. 13 becomes

$$[\mathbf{c}] = [\mathbf{X}]^{-1}[\mathbf{U}][\mathbf{m}] \tag{18}$$

Furthermore, the square matrix $[\mathbf{X}]$ given by Eq. 7 can also be represented for computational purposes in terms of $[\mathbf{m}]$ and $[\mathbf{U}]$

$$[\mathbf{X}] = [\mathbf{U}][\mathbf{m}][\mathbf{m}]^T[\mathbf{U}]^T \tag{19}$$

where the superscript T indicates the transpose of the matrix. (Thus, $[\mathbf{m}]^T$ has $2M+1$ rows and $R+1$ columns with elements given by $[[\mathbf{m}]^T]_{mn} = [\mathbf{m}]_{nm}$, and $[\mathbf{U}]^T = [\mathbf{U}]$ because it is square diagonal.)

The inverse matrix $[\mathbf{X}]^{-1}$ can then be represented

$$[\mathbf{X}]^{-1} = [\mathbf{U}]^{-1}[[\mathbf{m}][\mathbf{m}^T]]^{-1}[\mathbf{U}]^{-1} \tag{20}$$

where the only nonzero elements of the inverse units matrix $[\mathbf{U}]^{-1}$ are on the diagonal and given by $[\mathbf{U}]_{nn}^{-1} = 1/\tau^n$.

Using this in Eqs. 18 and 9 gives the finished form for the filter coefficients

$$[\mathbf{c}] = [\mathbf{U}]^{-1}[[\mathbf{m}][\mathbf{m}^T]]^{-1}[\mathbf{m}] \tag{21}$$

and the covariance matrix

$$[\boldsymbol{\sigma}_a^2] = \sigma_y^2[\mathbf{U}]^{-1}[[\mathbf{m}][\mathbf{m}^T]]^{-1}[\mathbf{U}]^{-1} \tag{22}$$

For our rotary encoder application, each raw y-value is an integer pulse count with each pulse representing an angular change of $\delta_\theta = 2\pi/1440$ rad. For determining the best estimates of θ, $d\theta/dt$, and $d^2\theta/dt^2$, these raw counts can be used with the conversion to radians also taken care of by applying the factor δ_θ to all Savitsky-Golay coefficients.

If the measurement probability distribution for the rotary count is assumed uniform with a width of $\pm 1/2$ a count, the standard deviation is $\sigma_y = \sqrt{1/12}$ counts. This is needed for the covariance matrix.

The LabVIEW programs for the Savitsky-Golay filtering are SavGolRaw.vi, which gives $[[\mathbf{m}][\mathbf{m}]^T]^{-1}[\mathbf{m}]$ and SavGolCoef.vi, which gives the zeroth, first, and second derivative coefficients, i.e., the first, second, and third row of $\delta_y[\mathbf{U}]^{-1}[[\mathbf{m}][\mathbf{m}]^T]^{-1}[\mathbf{m}]$, with the third row multiplied by 2.

The Excel spreadsheet SG.xls shows a table and graph for the 33-point quartic polynomial filters for the polynomial coefficients. It also gives the covariance matrix for the filter coefficients and the uncertainties in the filtered y, dy/dt, and d^2y/dt^2.