Nonlinear Dynamics On The Cheap In The Junior Laboratory

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Abstract: This past spring (2015) three of us (RK, RT, JC) team-taught a Junior level laboratory in which physics majors are required, over two semesters, to complete experiments in 8 out of 10 main topic areas of physics. As of the beginning of the spring 2015 semester, a satisfactory Mechanics/Nonlinear Dynamics experiment supplied with instructions had not been identified. Towards the end of the semester two Metropolitan State University of Denver students (NH, JZ) expressed an interest in investigating the nonlinear dynamics of Duffing's oscillator. Below, we describe the results of their efforts.

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INTRODUCTION

Nonlinear dynamics has applications in many areas of science, engineering, economics, and even music. With wide-ranging applications, it is important to understand the physical implications of nonlinear systems and the methods used to describe their behavior. Nonlinear systems defy description by the methods we can easily apply to linear systems. They do not follow the principle of superposition, so we cannot use the same methods to solve the differential equations that model their behavior. And, they have multiple equilibrium positions which are reached from different initial conditions. Also, self-sustained oscillations can arise as limit cycles in many nonlinear systems. When driven by a sinusoidal function, the nonlinear system will have an output signal that contains multiple harmonics and sub-harmonics of the driving signal with different phases. Perhaps the most important property of nonlinear systems is that they can exhibit chaos. Once chaotic motion has begun, the system will never come to equilibrium.

The Duffing oscillator is a classic system for analyzing nonlinear dynamics. It does not refer to a specific apparatus or experiment but is a famous mathematical model used to describe a damped, forced oscillator. In its most basic form the second order differential equation^{1,2} is:

$$\ddot{\theta} + \delta\dot{\theta} + \beta\theta + \alpha\theta^3 = \gamma \cos\omega t. \quad (1)$$

The amount of damping on the system is denoted by δ . α signifies the amount of nonlinearity in the restoring force. When $\alpha = 0$, and $\beta > 0$ the above equation describes a damped, driven simple harmonic oscillator. The amplitude of the periodic driving force is given by γ . The frequency of the periodic driving force is represented by ω . If $\gamma = 0$ we have a system without a driving force.

A DUFFING'S OSCILLATOR

The Duffing oscillator constructed for this experiment consisted of an inverted pendulum driven by an alternating magnetic field generated by two Helmholtz coils. The arm of the pendulum was made with a flexible stainless steel cable tie (a hacksaw blade was too stiff), and was attached to a Pasco Rotary Sensor and supported by springs that could be adjusted to balance the pendulum arm in an upright position, thereby, providing a double-well potential for nonlinear oscillations. Two neodymium magnets were attached near the top of the pendulum arm both to serve as the pendulum bob and to drive the pendulum's motion between the Helmholtz coils. The Helmholtz coils were driven by a Pasco Function Generator. The magnets were positioned so that the pendulum would perform small oscillations about the equilibrium points on either side. The balancing springs were adjusted to keep the two equilibrium positions close to being symmetrically spaced about the centerline.

In order to align the plane of the permanent magnets perpendicular to the field of the Helmholtz

coils the stainless steel cable tie was secured to the rotary sensor through a small hook at the bottom of the cable tie and a small screw was tightened to the post of the rotary sensor. Then, the two magnets were placed north-to-south on the cable tie and secured perpendicular to the magnetic field of the coils. The springs were attached by taping them to a cardboard cross piece taped to the bottom of the flexible metal cable tie. Fig. 1 is a picture of the apparatus.

FIGURE 1. Duffing apparatus showing the Helmholtz coils, Pasco Rotary Sensor and Pasco Function Generator. Springs attached on each side of the inverted cable tie are secured to ring stands to prevent the cable tie from falling over and to create a double-well potential.

Data acquisition was done with Data Studio (Pasco). Numerical analysis was performed in SciLab³ (open source software) on a desktop computer.

NOVEL DESIGN

This apparatus is not the typical design in which a thin beam (hacksaw blade, perhaps) is clamped at the bottom, weighted at the top, with permanent magnets placed near the bottom of the beam and then driven by nearby coils².

As revealed in the Duffing model of Eq. (1) the requirements for nonlinear oscillations are few: 1) a second-order system - like an oscillating beam that has nonlinear stiffness and 2) periodic forcing that creates a phase space with the minimal number of degrees of freedom (3 dimensions) that can exhibit chaos.

That said, the constraint of our particular choice of measurement instrument, the rotary sensor, required substantial modification of the traditional earlier realizations of a Duffing oscillator^{1,2,4}. Instead of a beam that is rigidly clamped at the base, the bottom of the beam must freely rotate around the axis of rotational position encoder. In order to prevent the beam from falling over, the students invented a spring tensioning system that allowed the undriven beam to only partially rotate to the left or right of vertical. Once tilted, the top-mounted weights caused the beam to bend as well. By using a pair of rare-earth magnets as the top weight, strong coupling to the oscillating magnetic field within a pair of Helmholtz coils provided a convenient method for periodic forcing. Fig. 2 is a schematic of the apparatus designed by NH and JZ.

FIGURE 2. Schematic of the Duffing oscillator showing the novel spring tensioning system used to create a double-well potential.

DATA COLLECTION

The Pasco Rotary Sensor was connected to a computer with DataStudio software installed. The position of the pendulum was recorded using DataStudio with a 1000 Hz sampling rate. The sinusoidal function generator was connected to the Helmholtz coils to drive the magnetic pendulum bob.

After carefully adjusting the springs (the supports at the far ends of the spring were moved in or out to change the spring tension) and adjusting the position

of the two magnets, the oscillator was driven at 1 to 2 Hz while DataStudio recorded the angular position of the pendulum. The pendulum system proved very sensitive to adjustments made to the springs and to the position of the magnets, often displaying drastically different motion after very small changes. When interesting chaotic motion or period doubling effects were observed, the apparatus was allowed to run for approximately 20 minutes while data was collected.

The position data for each run of the oscillator collected in DataStudio was saved as a text file and imported into SciLab as a two-column vector, with one column listing the sampling times and the other column listing the associated pendulum angular positions. A program was written in SciLab to select time and angular position entries from the imported data to calculate angular velocity and plot time series and phase-space diagrams. DataStudio proved to be insufficient for calculating velocity at the 1000 Hz sampling rate that was used. The data entries were selected at 0.025 sec intervals and the velocity was calculated using a time-centered average change in position with a time step of 0.05 sec for this purpose.

ANALYSIS

A key physical idea explored in this experiment is the chaotic motion of a damped, driven oscillator. Chaos is characterized by unpredictable, non-periodic motion, and by extreme sensitivity to initial conditions. In the Duffing oscillator constructed for this experiment, driving frequencies of 1.2 and 1.3 Hz induced non-periodic motion, and the modification of any initial conditions, such as the position of the magnets or the tension in the balancing springs, resulted in significantly different patterns of motion. The time series shown in Fig. 3 gives an example of the chaotic motion observed in the oscillator at 1.2 Hz.

As mentioned above, SciLab was used in the analysis of the experimental data by calculating angular velocity and generating plots of the motion. SciLab proved to be much better than Microsoft Excel in handling the more than 1.2 million data points in a typical data run.

An important idea that makes the experiment work is the existence of two potential wells of the Duffing oscillator which allows the pendulum to oscillate about two stable equilibrium points. When driven at the appropriate frequency the pendulum is able to switch back and forth between oscillations about these two equilibria. The ability of the pendulum to jump from one potential well to the other allows for the possibility of non-periodic motion as certain driving frequencies

produce patterns of motion that jump between the two wells unpredictably. This behavior is revealed in the phase-space diagram shown in Fig. 4 where the minima are clearly visible.

FIGURE 3. Experimental time series for the Duffing oscillator driven at 1.2 Hz.

FIGURE 4. Experimental phase-space for the Duffing oscillator driven at 1.2 Hz. This graph shows evidence of the double-well potential created by the spring tensioning system.

SciLab was also used to numerically model the Duffing equation of motion using parameters known to produce chaotic behavior. Time series, phase-space, and Poincare section plots were generated to compare against experimental results. A simple Euler-Cromer algorithm was used to generate the simulated data⁵.

An example of a phase space plot generated for the numerically simulated Duffing oscillator using SciLab is shown in Fig. 5.

FIGURE 5. Simulated phase-space for the Duffing oscillator plainly shows the existence of the double-well potential.

Clearly, the simulated data are much cleaner than the experimental data shown in Fig. 4. In either case the two minima are definitely distinguishable.

Another important approach to analyzing the data in this experiment was generating Poincare sections for the oscillator. Similar to the phase space diagram, the Poincare section is also a plot of the angular velocity of the pendulum versus its angular position, but only one point is plotted for every period of the driving force. In this way, certain features of the long-term behavior of the oscillator become clear.

Period-doubling effects can be observed by the clustering of points in the Poincare section; two distinct groupings would correspond to period-two motion, four distinct groupings correspond to period-four motion, and so on. Fig. 6 shows an example of period-six motion. This occurred at a driving frequency of 1.4 Hz.

FIGURE 6. Experimental Poincare section of the Duffing oscillator showing period-six motion at a driving frequency of 1.4 Hz.

Chaotic motion can be observed by a lack of distinct clustering, and is accompanied by the emergence of a shape called a strange attractor, over which the points are spread relatively uniformly. An example of a strange attractor that corresponds to the chaotic motion of the pendulum for a frequency of 1.2 Hz is shown in Fig. 7.

FIGURE 7. Experimental Poincare section for the Duffing oscillator driven at 1.2 Hz shows formation of a strange attractor.

DISCUSSION

This novel version of a driven, inverted, flexible pendulum provides a simple mechanism by which chaotic behavior can easily be produced and measured. The phase-space diagrams and Poincare sections show that the motion measured was truly chaotic at 1.2 Hz. The apparatus is cheap and easy to build from parts found in most undergraduate physics laboratories.

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