Using Split-Ring Resonators to Measure Complex Permittivity and Permeability

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Abstract: This paper describes how to measure the complex permittivity and permeability of materials using split-ring resonators. High- Q split-ring resonators, which can be modelled as LRC circuits, are easy to fabricate and can be used to measure electromagnetic material properties at frequencies that span approximately 10 MHz to 2 GHz. If the resonator is submerged in a liquid/suspension, its resonant frequency and quality factor will be modified from the in-air values by factors that depend on the relative permittivity and permeability of the liquid/suspension. General expressions for the resonator's frequency response are derived. Unlike an LRC circuit, the resonator's response is not strictly Lorentzian. However, a wide variety of cases for which the response is approximately Lorentzian are explored. For each of these cases it is demonstrated that the real and imaginary components of the relative permittivity and permeability can be extracted from the in-air and in-liquid/suspension resonant frequencies and quality factors.

Keywords: Split-ring resonator, Permittivity, Permeability, Quality factor

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I. INTRODUCTION

Split-ring resonators (SRRs) enjoy a wide variety of uses in current experimental physics research. They are key components of engineered metamaterials¹, used as tools for measuring the electromagnetic (EM) properties of materials²⁻⁵, and used as devices in novel microwave instruments⁶. Consisting of a conducting tube with a slit along its length (Fig. 1), SRRs that operate from approximately 10 to 2000 MHz are easy to fabricate^{5,7}.

This paper describes how SRRs can be used to measure both the real and imaginary components of the permittivity and permeability of materials or suspensions. This work is an extension of Ref. 5 in which a SRR was used to measure the dielectric constant and conductivity of saltwater solutions with varying concentrations of salt.

The paper is organized as follows: Section II reviews the properties of SRRs when suspended in air. Section III examines how the SRR properties are modified when submerged in a liquid with a complex permittivity. It is shown that, under many circumstances, the SRR response is approximately Lorentzian. Several real materials are considered to demonstrate how, in principle, the real and imaginary components of the permittivity can be extracted from successive measurements of the in-air and in-liquid resonant frequencies and quality factors. Section IV considers the case of a SRR submerged in a suspension of magnetic particles and shows how the real and imaginary components of the permeability can be extracted. The results are summarized in Sec. V.

II. SUSPENDED IN AIR

A split-ring resonator can be modelled as an LRC -series circuit⁷. The slit is modelled as a capacitor with $C_0 \approx \epsilon_0 w \ell / t$ and the bore as an inductor with $L_0 \approx \mu_0 \pi r_0^2 / \ell$. The resulting resonance frequency is

FIG. 1. (a) A schematic drawing of a SRR. (b) Cross-sectional view. The SRR has length ℓ , inner radius r_0 , wall thickness w , and a slit along its length of height t . In this work, dimensions that match the SRR reported in Ref. 5 were used in all calculations. Namely, $\ell = 4$ in. (10.2 cm), $r_0 = 11/16$ in. (1.75 cm), $w = 5/16$ in. (0.79 cm), and $t = 0.010$ in. (0.25 mm).

given by^{5,7}:

$$\omega_0 = \frac{1}{\sqrt{L_0 C_0}} \approx \frac{c}{r_0} \sqrt{\frac{t}{\pi w}} \quad (1)$$

where $c = 1/\sqrt{\epsilon_0 \mu_0}$ is the speed of light.

The resistance of the SRR determines the width of the resonance. Currents are restricted to be within a length $\delta = \sqrt{2\rho/(\mu_0 \omega)}$ of the inner surface of the SRR bore where δ is the EM skin depth. As a result, the effective resistance of SRR is given by⁵:

$$R(\omega) = \rho \frac{2\pi r_0}{\ell \delta(\omega)} \propto \sqrt{\omega} \quad (2)$$

where the explicit dependence on frequency has been indicated for later use. For a SRR made from a good conductor, ω_0 is expected to be much greater than the width of the resonance such that, over the frequency range of interest, the SRR resistance can be approximated as $R(\omega_0) \equiv R_0$ which leads to a simple expression for the expected quality factor $Q_0^{-1} = R_0 \sqrt{C_0/L_0} = \delta/r_0$. An aluminum SRR made using the dimensions given in Fig. 1 results in $Q_0 > 3000$ which justifies approximating $R(\omega)$

as R_0 . In practice, because the SRR experiences radiative losses, quality factors that approach the value predicted by Q_0 are achieved only if the resonator is surrounded by a larger cylindrical EM shield^{5,7}.

The detected SRR signal is proportional to the magnitude of the induced current and, therefore, inversely proportional to the magnitude of the effective impedance Z_0 which has a Lorentzian line shape given by:

$$|Z_0(\omega)| = \left[R_0^2 + \left(\omega L_0 - \frac{1}{\omega C_0} \right)^2 \right]^{1/2} \quad (3)$$

$$= R_0 Q_0 \left[\frac{1}{Q_0^2} + \left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \right)^2 \right]^{1/2} \quad (4)$$

where the second equation has been obtained using the substitutions $C_0^{-1} = \omega_0 R_0 Q_0$ and $L_0 = R_0 Q_0 / \omega_0$. Fits to the frequency dependence of $|Z_0(\omega)|^{-1}$ allow ω_0 and Q_0 to be extracted from the measured in-air SRR signal⁵.

III. RELATIVE PERMITTIVITY

Now imagine submerging the SRR in a liquid having a relative permittivity $\varepsilon_r = \varepsilon' - j\varepsilon''$. The liquid fills the gap of the SRR where the electric field is concentrated resulting in an enhanced and complex effective capacitance $C_1 = \varepsilon_r C_0$ and a modified resonance frequency ω_1 . The inductance is unaffected by the permittivity. However, recall from Eq. 2 that the SRR resistance is proportional to $\sqrt{\omega}$ such that the SRR impedance becomes:

$$Z_1(\omega) = R_0 \sqrt{\frac{\omega}{\omega_0}} + j\omega L_0 + \frac{1}{j\omega \varepsilon_r(\omega) C_0} \quad (5)$$

$$= \left[R_0 \sqrt{\frac{\omega}{\omega_0}} + \frac{\varepsilon''}{|\varepsilon_r|^2} \frac{1}{\omega C_0} \right] + j \left[\omega L_0 - \frac{\varepsilon'}{|\varepsilon_r|^2} \frac{1}{\omega C_0} \right] \quad (6)$$

where $|\varepsilon_r|^2 \equiv (\varepsilon')^2 + (\varepsilon'')^2$. Notice that a complex ε_r leads to an additional real term in Z_1 and hence additional losses. If C_0 and L_0 are once again eliminated in favour of ω_0 and Q_0 the result is:

$$\frac{|Z_1(\omega)|}{R_0 Q_0} = \left\{ \left[\frac{1}{Q_0} \sqrt{\frac{\omega}{\omega_0}} + \frac{\varepsilon''(\omega)}{|\varepsilon_r(\omega)|^2} \frac{\omega_0}{\omega} \right]^2 + \left[\frac{\omega}{\omega_0} - \frac{\varepsilon'(\omega)}{|\varepsilon_r(\omega)|^2} \frac{\omega_0}{\omega} \right]^2 \right\}^{1/2} \quad (7)$$

where the fact that ε_r is frequency dependent has been explicitly indicated. Although the general result given by Eq. 7 exhibits resonant behaviour, it is non-Lorentzian because $\Re[Z_1]$ is not frequency independent and because of the $\varepsilon_r(\omega)$ frequency dependence in $\Im[Z_1]$. The SRR response can be fit to the frequency dependence of $|Z_1(\omega)|^{-1}$ to extract experimental values of ε' and ε'' near the resonance frequency ω_1 .

Alternatively, if the width of the resonance is much less than ω_1 and $\varepsilon_r(\omega)$ does not change too rapidly, the

FIG. 2. Comparison of the exact $R_0/|Z_1|$ (Eq. 7, solid line) to the approximate Lorentzian curve (dashed line) with Teflon ($\varepsilon_r = 2.2$) filling the gap of the SRR. All calculations done assuming an aluminum ($\rho = 2.82 \times 10^{-8} \Omega^{-1}\text{m}^{-1}$) SRR made using the dimensions given in Fig. 1.

shape of the SRR response is approximately Lorentzian with ω_1 and the quality factor determined from:

$$\omega_1 \approx \frac{\sqrt{\varepsilon'(\omega_1)}}{|\varepsilon_r(\omega_1)|} \omega_0 \quad (8)$$

$$\frac{1}{Q_1} \approx \frac{1}{Q_0} \left[\frac{|\varepsilon_r(\omega_1)|^2}{\varepsilon'(\omega_1)} \right]^{1/4} + \frac{\varepsilon''(\omega_1)}{\varepsilon'(\omega_1)}. \quad (9)$$

The following sections show how the SRR can be used to extract ε' and ε'' from various materials. Three cases are explored: A) the Q_0^{-1} term in Eq. 9 dominates, B) the ε'' term dominates, and C) the two terms are comparable.

A. Q_0^{-1} term dominant

If the Q_0^{-1} term in Eq. 9 is dominant, then ε'' is negligible and $\varepsilon_r \approx \varepsilon'$ such that $\omega_1 \approx \omega_0 / \sqrt{\varepsilon'}$ and $Q_1 \approx Q_0 \sqrt[4]{\varepsilon'}$. As an example, consider inserting a tight-fitting piece of Teflon, for which $\varepsilon_r = 2.2$, into the gap of the SRR. Figure 2 compares the exact $R_0/|Z_1|$ calculated using Eq. 7 to the Lorentzian approximation obtained by replacing ω_0 and Q_0 in Eq. 4 by the ω_1 and Q_1 expressions given above and in the first row of Table I. The approximate Lorentzian is indistinguishable from the exact curve and, as shown in Table I, experimental values of ε' can be determined from either $(\omega_0/\omega_1)^2$ or $(Q_0/Q_1)^4$.

B. ε'' term dominant

The opposite extreme occurs when the ε'' term dominates Eq. 9. In this case, ω_1 is determined from Eq. 8 and $Q_1 \approx \varepsilon'/\varepsilon''$. To numerically solve Eq. 8 for ω_1 , the

FIG. 3. The complex relative permittivity of 2-propanol⁹ and water¹⁰ as a function of frequency.

FIG. 4. Comparison of the exact $R_0/|Z_1|$ (Eq. 7, solid line) to the approximate Lorentzian curve (dashed line) with the SRR submerged in 2-propanol.

frequency dependencies of the real and imaginary components of ϵ_r must be known. As an example, Fig. 3 shows the relative permittivity of 2-propanol⁸ as a function of frequency⁹. With these assumed frequency dependencies, Eq. 8 yields $\omega_1/(2\pi) = 63.3$ MHz and $Q_1^{-1} \approx 0.1 \gg Q_0^{-1}$. Figure 4 shows $R_0/|Z_1|$ calculated using the exact expression given by Eq. 7 and the approximate Lorentzian curve obtained using the ω_1 and Q_1 expressions in the second row of Table I. Once again, the exact and approximate curves are, for all practical purposes, indistinguishable. Table I shows how measurements of the in-air and in-liquid resonance frequencies and quality factors can be used to experimentally determine ϵ' and ϵ'' .

C. Q_0^{-1} & ϵ'' terms comparable

Finally, water is used as an example of a material for which the Q_0^{-1} and ϵ'' terms are comparable. The permit-

FIG. 5. Comparison of the exact $R_0/|Z_1|$ (Eq. 7, solid line) to the Lorentzian curves (dashed lines) with the SRR submerged in water. The three Lorentzians were generated using $Q_1 = Q_0 (\epsilon')^{-1/4}$, $Q_1 = \epsilon'/\epsilon''$, and Q_1 given by Eq. 9. The vertical axis is shown on a log scale to emphasize the differences between the three Lorentzian curves.

tivity of water is shown in Fig. 3¹⁰. The resonance frequency obtained from Eq. 8 is $\omega_1/(2\pi) = 31.2$ MHz. Figure 5 compares the exact form of $R_0/|Z_1|$ to Lorentzian curves obtained using $Q_1 = Q_0 (\epsilon')^{-1/4}$, $Q_1 = \epsilon'/\epsilon''$, and Q_1 as given in Eq. 9. In this case, the full expression for Q_1 is required to accurately reproduce the frequency dependence of $|Z_1|^{-1}$. Table I shows how the real and imaginary components of ϵ_r can be extracted from in-air and in-liquid measurements of the SRR resonance frequency and quality factor.

IV. RELATIVE PERMEABILITY

The same type of analysis presented in Sec. III can be repeated for the case of a SRR submerged in a material having a relative permeability $\mu_r = \mu' - j\mu''$. In this case, the inductance $L_2 = \mu_r L_0$ is modified such that:

$$\frac{|Z_2(\omega)|}{R_0 Q_0} = \left\{ \left[\frac{1}{Q_0} \sqrt{\frac{\omega}{\omega_0}} + \mu'' \frac{\omega}{\omega_0} \right]^2 + \left[\mu' \frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \right]^2 \right\}^{1/2} \quad (10)$$

which, again, is a non-Lorentzian response. Of course, if the modified resonance frequency ω_2 is large compared to the width of the resonance, an approximately Lorentzian response is recovered. The relationships between ω_2 , Q_2 , μ' , and μ'' for the various cases (Q_0^{-1} dominant, μ'' dominant, and the general case in which the two terms are comparable) are summarized in the bottom half of Table I.

Because of its geometry, it is only practical to use SRRs to measure the permeability of magnetic particles suspended in a liquid. One would first measure the resonance of the SRR submerged in the liquid without any

Permittivity	ω_1	Q_1	$\epsilon'(\omega_1)$	$\epsilon''(\omega_1)$
Q_0^{-1} dominant	$\frac{\omega_0}{\sqrt{\epsilon'}}$	$\frac{Q_0}{(\epsilon')^{1/4}}$	$\left(\frac{\omega_0}{\omega_1}\right)^2$ and $\left(\frac{Q_0}{Q_1}\right)^4$	—
ϵ'' dominant	$\omega_0 \frac{\sqrt{\epsilon'}}{ \epsilon_r }$	$\frac{\epsilon'}{\epsilon''}$	$\frac{\left(Q_1 \frac{\omega_0}{\omega_1}\right)^2}{1 + Q_1^2}$	$\frac{Q_1 \left(\frac{\omega_0}{\omega_1}\right)^2}{1 + Q_1^2}$
general	$\omega_0 \frac{\sqrt{\epsilon'}}{ \epsilon_r }$	$\left[\frac{1}{Q_0} \left(\frac{ \epsilon_r ^2}{\epsilon'} \right)^{1/4} + \frac{\epsilon''}{\epsilon'} \right]^{-1}$	$\frac{\left(\frac{\omega_0}{\omega_1}\right)^2}{1 + \left[\frac{1}{Q_1} - \frac{1}{Q_0} \left(\frac{\omega_0}{\omega_1}\right)^{1/2} \right]}$	$\frac{\left(\frac{\omega_0}{\omega_1}\right)^2 \left[\frac{1}{Q_1} - \frac{1}{Q_0} \left(\frac{\omega_0}{\omega_1}\right)^{1/2} \right]}{1 + \left[\frac{1}{Q_1} - \frac{1}{Q_0} \left(\frac{\omega_0}{\omega_1}\right)^{1/2} \right]}$

Permeability	ω_2	Q_2	$\mu'(\omega_2)$	$\mu''(\omega_2)$
Q_0^{-1} dominant	$\frac{\omega_0}{\sqrt{\mu'}}$	$Q_0 (\mu')^{3/4}$	$\left(\frac{\omega_0}{\omega_2}\right)^2$ and $\left(\frac{Q_2}{Q_0}\right)^{4/3}$	—
μ'' dominant	$\frac{\omega_0}{\sqrt{\mu'}}$	$\frac{\mu'}{\mu''}$	$\left(\frac{\omega_0}{\omega_2}\right)^2$	$\frac{1}{Q_2} \left(\frac{\omega_0}{\omega_2}\right)^2$
general	$\frac{\omega_0}{\sqrt{\mu'}}$	$\left[\frac{1}{Q_0} \frac{1}{(\mu')^{3/4}} + \frac{\mu''}{\mu'} \right]^{-1}$	$\left(\frac{\omega_0}{\omega_2}\right)^2$	$\left(\frac{\omega_0}{\omega_2}\right)^2 \left[\frac{1}{Q_2} - \frac{1}{Q_0} \left(\frac{\omega_2}{\omega_0}\right)^{3/2} \right]$

TABLE I. Top: Summary of the relationships between ω_0 , Q_0 , ω_1 , Q_1 and the complex relative permittivity. Bottom: Summary of the relationships between ω_0 , Q_0 , ω_2 , Q_2 and the complex relative permeability.

magnetic particles present to determine ω_0 and Q_0 . Next, the resonance would be remeasured with the magnetic particles suspended in the liquid to find ω_2 and Q_2 . Finally, μ' and μ'' could then be determined using the appropriate relationships given in Table I.

V. SUMMARY

Methods for determining the complex relative permittivities and permeabilities of materials using a SRR have

been described. In general, the SRR signal response is non-Lorentzian. However, in a wide variety of cases the signal is very accurately approximated by a Lorentzian. For dielectric materials, the limiting cases of $Q_0^{-1} \gg \epsilon''/\epsilon'$ and $Q_0^{-1} \ll \epsilon''/\epsilon'$ as well as the general case $Q_0^{-1} \sim \epsilon''/\epsilon'$ were all considered. Teflon, 2-propanol, and water were used as examples of these three cases and it was shown how measurements of the in-air and in-liquid resonant frequencies and quality factors can be used to experimentally determine ϵ' and ϵ'' . An equivalent analysis for the case of a SRR submerged in a ferromagnetic suspension was also presented.

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