

# A Lab for Exploring the Precession and Nutation of a Gyroscope

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## 1. Introduction

A gyroscope is a device which spins and can be used to measure orientation using the principle of angular momentum conservation. Practical and usually very precise versions of such a device are used in navigation, both naval, aircraft and spacecraft.

The simplest example of an object whose motion is based on the same principles is the heavy spinning top. The motion of the toy top has been the source of fascination for ages (planet Earth is also a top!). Both children and adults (see photos below) have delighted in observing and playing with this counterintuitive toy. The sense of wonder which it inspires is due, in large part, to our lack of familiarity with rotational motion compared to translational motion.



Hmong tribe boy playing with homemade top (left); two adults (W. Pauli and N. Bohr) playing with a “tippy-top” (right)



## 2. Motivation

The beginning physics or engineering student has limited theoretical and experimental exposure to gyroscopic motion. Usually this consists of an in-class demo, e.g. bicycle wheel along with the derivation of the simplest torque induced steady-precession model. Discussion of nutation is either omitted or brief and qualitative. At the intermediate<sup>1,2</sup> or advanced<sup>3</sup> mechanics level, there may be theoretical treatment of the heavy top (although time constraints most often prevent that) but rarely, if ever, is there a matching laboratory exercise.

In this work we have attempted to demonstrate that by applying small modifications to off-the-shelf equipment and available program code, a fairly simple lab experiment can

be conducted which accurately measures the torque-induced motion of a heavy spinning top and compares it to a detailed model, accounting for both precession and nutation, based on the Euler-Lagrange equations of motion. An instructor of an intermediate to advanced mechanics course may choose (time permitting) to incorporate a laboratory project similar to this.

### 3. Theory and Numerical Simulation

#### A) Steady precession model:

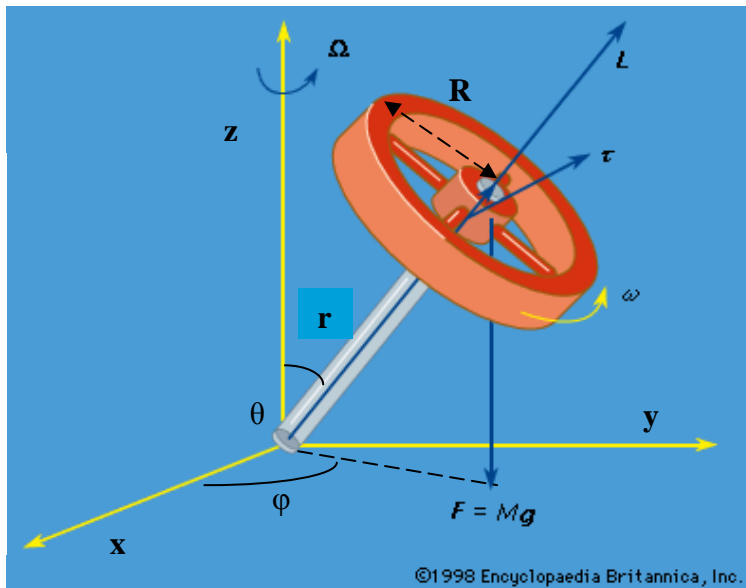
*“Science may be described as the art of systematic over-simplification.”*

Carl Popper

This is the basic treatment which neglects nutation and assumes a constant precession velocity. The well known simple relation is:

$$(1) \quad \Omega = \frac{MgR}{\lambda_3 \omega} \quad ,$$

**Figure 1**  
Basic parameters of the spinning top  
(from Encyclopaedia Britannica)



where  $\Omega$  is the precession speed,  $M$ - the mass of the disk,  $R$ - the radius of the disk,  $\omega$ - the angular spin velocity, and  $\lambda_3$ - the moment of inertia about the axis of rotation. We have used this as a starting point in approximating to the motion of the top. From several  $\Omega$  vs.  $\omega$  sets (Fig. 2 a, 2b) we have extracted an effective value for  $\lambda_3$  which compares well to the theoretical value of an ideal thick disk spinning on a massless rod.

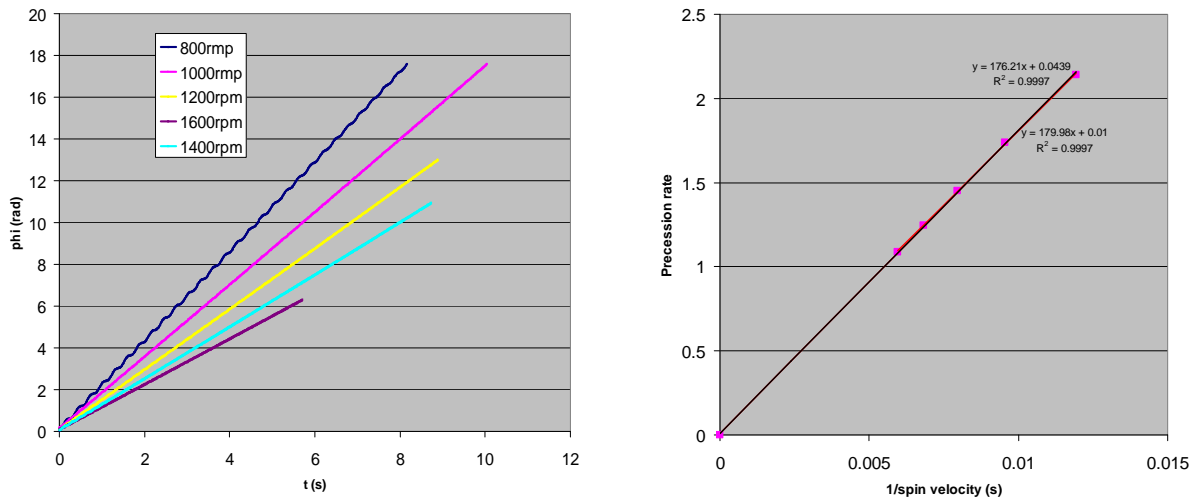


Figure 2

a)  $\phi(t)$  vs.  $t$  plots at different rpm value

b) Linear fit to extract  $\lambda_3$

### B) Exact model of symmetric, frictionless top:

*“Things should be made as simple as possible, but not any simpler.”*

Albert Einstein

In order to include nutation in the model and treat it quantitatively, one must begin with the exact equations of motion. These are most easily obtained and treated in the Lagrangian formulation. A simplifying assumption is that the top is treated as a perfectly symmetric rotor, i.e., an ideal *thick uniform disk rotating about a massless rod*. The Lagrangian in spherical coordinates is

$$(2) \quad L = T - U$$

where

$$(3) \quad T = \frac{1}{2} \lambda_1 \left( \dot{\phi}^2 \sin^2 \theta + \dot{\theta}^2 \right) + \frac{1}{2} \lambda_3 \left( \dot{\psi} + \dot{\phi} \cos \theta \right)^2$$

is the kinetic energy and

$$(4) \quad U = MgR \cos \theta$$

is the potential energy. The Euler-Lagrange (EL) equations are

$$(5) \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0, i = 1, 2, 3$$

where

$$(6) \quad q_1 = \varphi, q_2 = \theta, q_3 = \psi$$

are the standard Euler angles. The first two are shown of Figure 1 and the third is interpreted as the angle of spin. The EL equations which are non-separable for the angular functions can be solved numerically<sup>4,5</sup>.

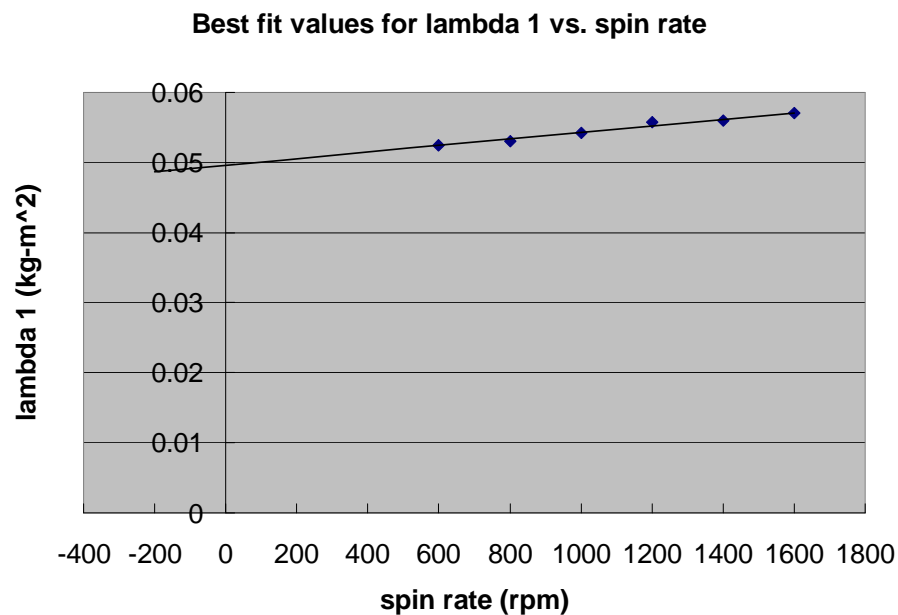
The procedure can be summarized as follows:

**Input:**  $M, R, r, \lambda_3$  (moment of inertia about axis of spin),  $\lambda_1$  (moment of inertia about a perpendicular axis), disk thickness and initial conditions- angles and angular speeds

**Solve:** system of coupled second order ordinary differential equations using:  
 a) computer algebra software: e.g., Maple or Mathematica (equally best)  
 b) scientific computation software: e.g., MATLAB (good)  
 c) programming language: e.g., C, Fortran (not recommended)

**Output:**  $\varphi(t), \theta(t), \psi(t)$

We would like to point out that for the purpose of obtaining the best fit of the theoretical curves to our experimental angular function data, the numerical value for  $\lambda_1$  was obtained as a best fit parameter. It compares well with what one would calculate using the parallel axis theorem and neglecting the mass of the rod, but is not constant and seems to show a slight dependency on the spin angular velocity of the disk (Fig. 3)



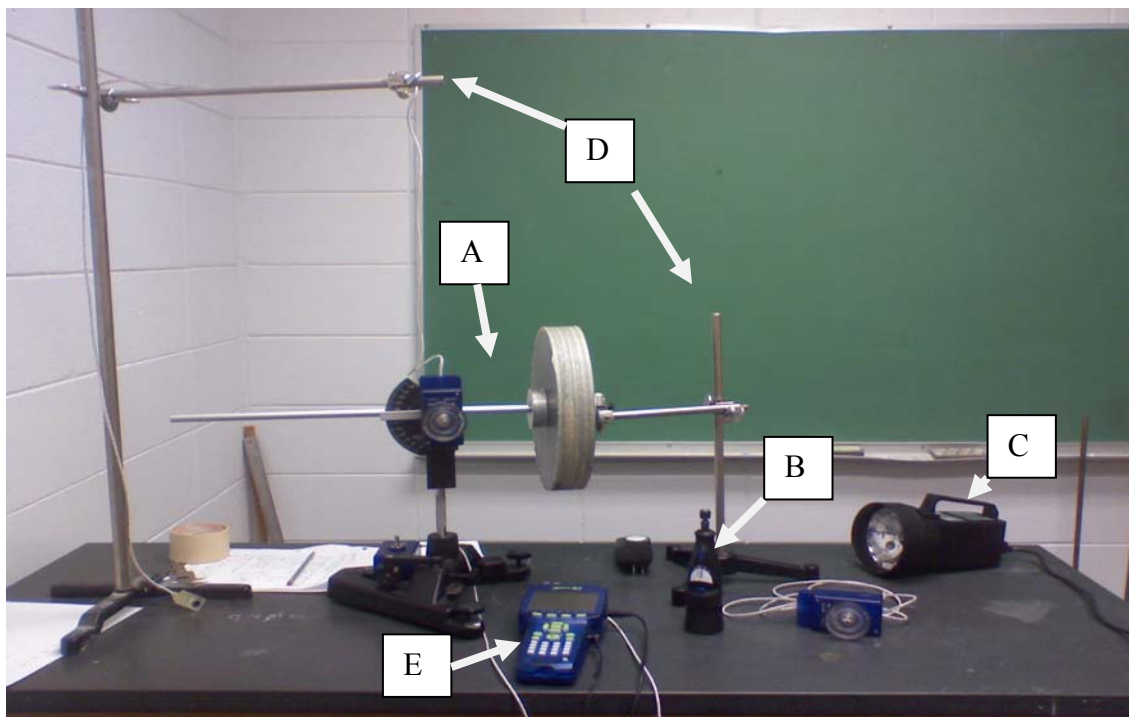
**Figure 3**

$\lambda_3$  shows a slight increase with rpm

#### 4. Experiment

The disk was accelerated beyond a target initial spin rate and released from rest when the aliasing pattern of a digital stroboscope showed coincidence in the fundamental frequency. This limits the experiment to the  $\dot{\varphi}(0) = 0$  regime. Simultaneous measurements of  $\varphi(t)$  and  $\theta(t)$  were recorded for several different initial spin rates, i.e., different  $\omega(0) = \dot{\psi}(0)$  values, and two different initial azimuth angles and disk masses. Same parameter data was averaged. At high spin rates (above 1000 rpm) the friction in the joint bearing between the spin axis rod and that of the stand rapidly attenuates the nutational motion and it asymptotically converges to the average (Fig. 5 d – 5 f)

The experimental setup is easy to put together. Because of the large energy of the spinning disk, some caution has to be observed in keeping the gyroscope securely attached to the table base (it's best to bolt the stand to the base which prevents toppling over). The disk itself has to be tested at high spin rates (up to 1600 rpm in our experiment) before allowing it to precess. The setup is shown in the photograph below.



**Figure 4** The experimental setup

- A. PASCO demonstration gyroscope with two digital rotational sensors
- B. Mini drill with rubber head to accelerate disk(s) to target rpm values
- C. Digital stroboscope to measure initial spin rate by observing aliasing pattern of spot on disk

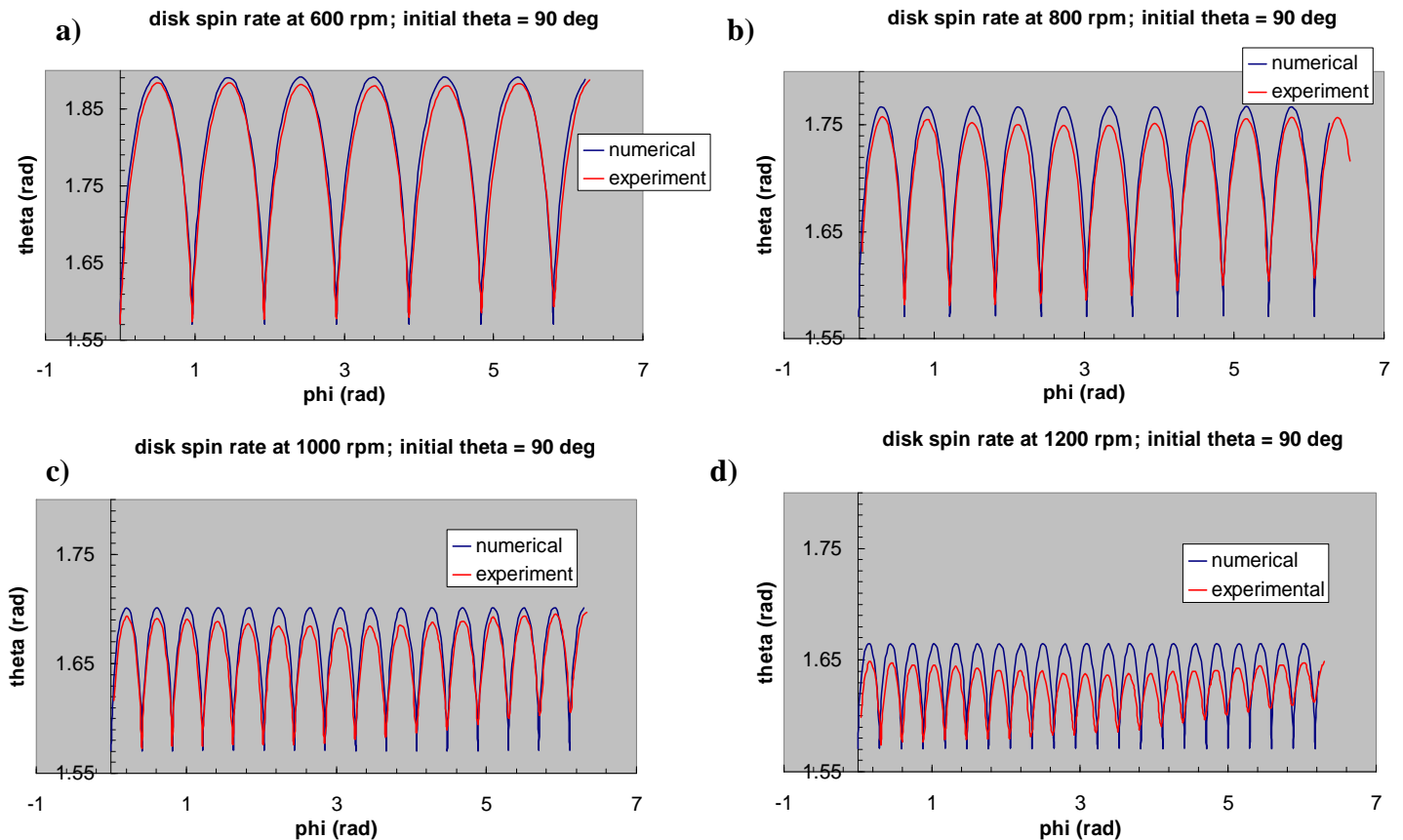
- D. Supporting stands
- E. PASCO Explorer data acquisition device

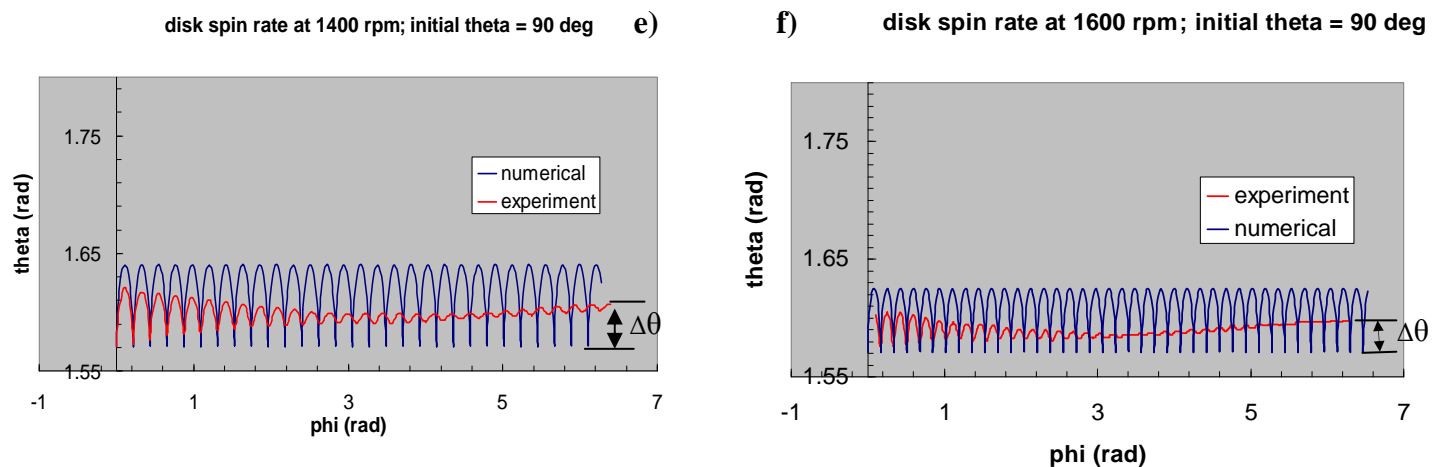
## 5. Results

Some of the main results obtained in this experiment are:

- The effective  $\lambda_3$  extracted from slope of a linear fit to the data (Fig. 2 b)) gives excellent agreement with theoretical value.
- The parameter  $\lambda_1$  obtained from a best-fit to the data (see 3 B) is in good agreement with the theoretical value although it shows a slight increase with rpm value (Fig.3)
- The theoretical model gave a good agreement with the experimental data for six different initial values of the disk spin velocity (Fig. 5a-5f) comprising a range from 600 rpm to 1600 rpm. In addition, a second  $\theta_0$  value (Fig. 6), and a second mass value of the spinning disk (Fig. 7) were tested and also gave good agreement.
- The measured asymptotic “dip” angle below the horizontal,  $\Delta\theta_0$ , (e.g. Fig. 5e and Fig. 5f) implies conservation of total angular momentum<sup>6</sup>, i.e., orbital (precession) plus spin, within the uncertainty of the measurement.

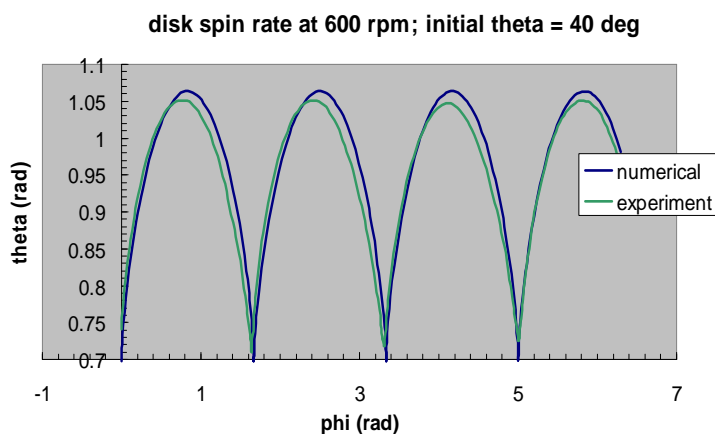
**Figure 5** Theoretical (fitted) vs. experimental motion curves at at  $\theta_0 = 0^\circ$





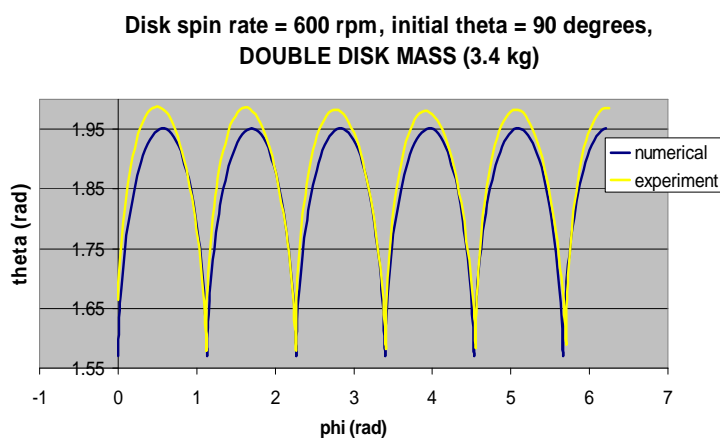
**Figure 6**

Motion curves for gyroscope started at  $\theta_0 = 40^\circ$



**Figure 7**

Motion curves for gyroscope started at  $\theta_0 = 0^\circ$  using two disks each of mass 1.7 kg



## 6. Summary

- A commercial demonstration gyroscope was adapted to accurately measure both precession and nutation at high spin rate of the disk.

- Numerical simulation of the data based on the perfectly symmetric and frictionless heavy top theoretical model was performed.
- Good to excellent agreement between data and model was found over a relatively large range of initial conditions of the dynamical system.
- Student lab projects based on this work could be valuable at both intermediate level (junior lab) and the more advanced (independent student research).

## 7. Suggestions for Follow-Up Work

- Modifications and improvements to the apparatus to further reduce friction
- Full error analysis
- Extend model to include non-conservative, i.e., frictional term(s) for closer agreement with experimental data (e.g., Rayleigh dissipative functions, to be empirically determined from data)
- Incorporate a best fit parameter analysis routine within the numerical algorithm

## 8. Acknowledgements

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We also thank Jason Klein for his steady hand and helpful discussion.

## 9. References

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